## A miscellany of results

Lemma 0. For monotonic $f$ and natural $n$, the extreme solutions of

$$X: [f^{n+1} X \equiv X] \qquad (0)$$

are independent of $n$.

Proof. Since $f^{n+1}$ is monotonic for all natural $n$, the extreme solutions exist for all natural $n$. Since the conjugate of $f$ is monotonic, it suffices to prove the lemma for the strongest solutions only.

We denote the strongest solution of (0) by $Q\,n$. Hence, for any natural $n$,

$$\begin{array}{l}
\quad true \\
= \{\text{definition of } Q\,0\} \\
\quad [f(Q\,0) \equiv Q\,0] \\
\Rightarrow \{\text{induction}\} \\
\quad [f^{n+1}(Q\,0) \equiv Q\,0] \\
\Rightarrow \{Q\,n \text{ being a strongest solution}\} \\
\quad [Q\,n \Rightarrow Q\,0]
\end{array}$$

Also, for any natural $n$,

$$\begin{array}{l}
\quad true \\
= \{\text{definition of } Q\,n\} \\
\quad [f^{n+1}(Q\,n) \equiv Q\,n] \\
\Rightarrow \{\text{predicate calculus}\} \\
\quad [(\underline{A}\,i: 0 \le i < n: f^{i+1}(Q\,n)) \wedge f^{n+1}(Q\,n) \equiv \\
\qquad Q\,n \wedge (\underline{A}\,i: 0 \le i < n: f^{i+1}(Q\,n))]
\end{array}$$

$$
\begin{array}{l}
= \{\text{predicate calculus}\} \\
\quad [(\underline{A}i : 0 \le i \le n : f^{1+i}(Qn)) \equiv (\underline{A}i : 0 \le i \le n : f^{i}(Qn))] \\
\Rightarrow \{\text{monotonicity of } f\} \\
\quad [f(\underline{A}i : 0 \le i \le n : f^{i}(Qn)) \Rightarrow (\underline{A}i : 0 \le i \le n : f^{i}(Qn))] \\
\Rightarrow \{Q0 \text{ being a strongest solution; Knaster-Tarski}\} \\
\quad [Q0 \Rightarrow (\underline{A}i : 0 \le i \le n : f^{i}(Qn))] \\
\Rightarrow \{\text{predicate calculus}\} \\
\quad [Q0 \Rightarrow f^{0}(Qn)] \\
= \{\text{functional iteration}\} \\
\quad [Q0 \Rightarrow Qn] \quad .
\end{array}
$$

(End of Proof.)

* * *

In EWD849a-2, we introduced different types of conjunctivity for predicate transformers. The following lemma connects three of them.

Lemma 1. For any predicate transformer $f$ we have

$$
\begin{array}{l}
(f \text{ is denumerably conjunctive}) \equiv \\
(f \text{ is conjunctive}) \wedge (f \text{ is } \underline{\text{and}}\text{-continuous}) \quad .
\end{array}
$$

Proof. The fact that the left-hand side implies both terms of the right-hand side has already been shown in EWD849a to follow immediately from the definitions. In order to prove the implication in the other direction, we derive, under the assumption of the right-hand side, that $f$ is denumerably conjunctive, i.e. for any sequence of predicates $Pi$ $(0 \le i)$

$$[f(\underline{A}i :: Pi) \equiv (\underline{A}i :: f(Pi))] \quad .$$

We observe for all $Z$

$$[Z \equiv f(\underline{A}i :: P_i)]$$
$= \{$predicate calculus$\}$
$$[Z \equiv f(\underline{A}j :: (\underline{A}i : 0 \leq i \leq j : P_i))]$$
$= \{f$ is and-continuous; $(\underline{A}i : 0 \leq i \leq j : P_i)$ is strengthening$\}$
$$[Z \equiv (\underline{A}j :: f(\underline{A}i : 0 \leq i \leq j : P_i))]$$
$= \{f$ conjunctive and range non-empty$\}$
$$[Z \equiv (\underline{A}j :: (\underline{A}i : 0 \leq i \leq j : f(P_i)))]$$
$= \{$predicate calculus$\}$
$$[Z \equiv (\underline{A}i :: f(P_i))]$$

(End of Proof.)

* * *

Lemma 2. For conjunctive $f$ and monotonic $g$ and predicate $B$ such that

$$[f X \equiv B \wedge g\ X] \text{ for all } X$$

we have $$[(\underline{A}i : i \geq 0 : f^i\ true) \equiv (\underline{A}i : i \geq 0 : g^i\ B)] \quad .$$

Proof. To begin with, we prove, using the conjunctivity of $f$, via mathematical induction for all natural $i$

$$[f^i\ true \equiv (\underline{A}j : 0 \leq j < i : g^j\ B) \wedge g^i\ true] \qquad .(1)$$

This holds for $i=0$. Furthermore we observe

$$[f^i\ true \equiv (\underline{A}j : 0 \leq j < i : g^j\ B) \wedge g^i\ true]$$
$\Rightarrow \{f$ is a function$\}$
$$[f^{i+1}\ true \equiv f((\underline{A}j : 0 \leq j < i : g^j\ B) \wedge g^i\ true)]$$
$= \{f$ is conjunctive, applied to finite, nonempty conjunction$\}$

$$[f^{i+1}\,true \equiv (\underline{A}j: 0 \le j < i: f(g^j B)) \wedge f(g^i\,true)]$$
$= \{[fX \equiv B \wedge gX]$ for all $X\}$
$$[f^{i+1}\,true \equiv B \wedge (\underline{A}j: 0 \le j < i: g^{j+1} B) \wedge g^{i+1}\,true]$$
$= \{[B \equiv g^0 B]\}$
$$[f^{i+1}\,true \equiv (\underline{A}j: 0 \le j < i+1: g^j B) \wedge g^{i+1}\,true]$$

which completes the proof of (1) for all natural $i$.
Hence

$$true$$
$= \{(1)$ and predicate calculus$\}$
$$[(\underline{A}i:: f^i\,true) \equiv (\underline{A}i:: g^i B) \wedge (\underline{A}i:: g^i\,true)]$$
$= \{g$, and hence $g^i$, is monotonic$\}$
$$[(\underline{A}i:: f^i\,true) \equiv (\underline{A}i:: g^i B)]$$

(End of Proof.)

<u>Peccavimus</u> In the proof of the similar formula (5) of EWD816, on line 7 of p.10, the motivation

$\{wlp(IF, ?)$ is conjunctive$\}$

is insufficient: we should have written "is universally conjunctive". (End of Peccavimus.)

<u>Lemma 3</u> For denumerably conjunctive $f$, monotonic $g$, and predicate $B$ such that

$$[fX \equiv B \wedge gX] \text{ for all } X$$

$(\underline{A}i:: g^i B)$ is the weakest solution of $X: [X \equiv fX]$.

<u>Proof</u>. Since $f$ is denumerably conjunctive, $f$ is <u>and</u>-continuous, and hence the weakest solution of $X: [X \equiv fX]$ is $(\underline{A}i:: f^i\,true)$. Since $f$ is con-

junctive, and $g$ is monotonic, Lemma 2 is applicable, which concludes the proof. (End of Proof.)

Note that, on account of Lemma 1, the requirement on $f$ in Lemma 3 is not stronger than necessary.

* * *

<u>Lemma 4</u> For a predicate transformer $f$ satisfying

$$[f X \wedge f(\neg X) \equiv f \text{ false}] \wedge [f X \vee f(\neg X) \equiv f \text{ true}]$$

for all $X$,

(f's type of conjunctivity) = (f's type of disjunctivity)

for all types except universal.

<u>Proof</u>. We observe for any $X$

$$[f X \wedge f(\neg X) \equiv f \text{ false}] \wedge [f X \vee f(\neg X) \equiv f \text{ true}]$$

$\Rightarrow$ {termwise conjunction with $[\neg f(\neg X) \equiv f^* X]$ in 2nd term}

$$[f X \wedge f(\neg X) \equiv f \text{ false}] \wedge [f X \wedge \neg f(\neg X) \equiv f^* X \wedge f \text{ true}]$$

$\Rightarrow$ {termwise disjunction}

$$[f X \equiv f \text{ false} \vee (f^* X \wedge f \text{ true})]$$

Since $f$ false and $f$ true are predicates independent of $X$, any junctivity of $f^*$, except universal, is inherited by the right-hand side, and hence by $f$. Since a con(dis)junctivity property of $f^*$ is the same dis(con)junctivity property of $f$, the Lemma follows. (End of Proof.)

* * *

Lemma 5. Let monotonic $f$ and predicate $Y$ be such that

$$X:[fY \Rrightarrow fX] \qquad (2)$$

has a strongest solution; this is then also the strongest solution of

$$X:[fY \equiv fX] \quad . \qquad (3)$$

Proof. Let $Z$ be the strongest solution of (2). We observe

$$
\begin{array}{l}
\quad \text{true} \\
= \{Y \text{ is a solution and } Z \text{ is the strongest solution of } (2)\} \\
\quad [Z \Rrightarrow Y] \\
\Rrightarrow \{f \text{ is monotonic}\} \\
\quad [fZ \Rrightarrow fY] \\
= \{Z \text{ is a solution of } (2)\} \\
\quad [fZ \Rrightarrow fY] \wedge [fY \Rrightarrow fZ] \\
= \{\text{predicate calculus}\} \\
\quad [fY \equiv fZ]
\end{array}
$$

hence $Z$ is a solution of (3). Since the solutions of (3) constitute a subset of those of (2), of which $Z$ is the strongest, $Z$ is the strongest solution of (3). (End of Proof.)

* * *

Lemma 6. For any predicate transformers $h$ and $k$, the equation — in predicate transformers! —

$$f:(\underline{A}X{::}\,[fX \equiv hX \vee f(kX)]) \qquad (4)$$

has a strongest solution $g$, defined by

$$[g\,X \equiv (\underline{E}i : 0 \leq i : h(k^i X))] \quad \text{for all } X \quad .$$

<u>Proof</u>. We observe for any $X$

$$
\begin{array}{ll}
 & true \\
= & \{\text{definition of } g\} \\
 & [g\,X \equiv (\underline{E}i : 0 \leq i : h(k^i X))] \\
= & \{\text{isolation of 1st term and transformation of the dummy}\} \\
 & [g\,X \equiv h\,X \vee (\underline{E}i : 0 \leq i : h(k^i(k\,X)))] \\
= & \{\text{definition of } g\} \\
 & [g\,X \equiv h\,X \vee g(k\,X)] \quad ,
\end{array}
$$

hence, $g$ is a solution of (4) .

In order to prove that $g$ is the strongest solution of (4), we observe for any solution $f$ of (4)

$$
\begin{array}{lll}
 & (\underline{A}X :: [g\,X \Rrightarrow f\,X]) & \\
= & \{\text{definition of } g\} & \\
 & (\underline{A}X :: [(\underline{E}i :: h(k^i X)) \Rrightarrow f\,X]) & \\
= & \{\text{predicate calculus}\} & \\
 & (\underline{A}i :: (\underline{A}X :: [h(k^i X) \Rrightarrow f\,X])) & (5)
\end{array}
$$

and we shall prove (5) by induction over $i$ .

For the base we observe

$$
\begin{array}{ll}
 & true \\
= & \{f \text{ is a solution of } (4)\} \\
 & (\underline{A}X :: [h\,X \vee f(k\,X) \equiv f\,X]) \\
\Rightarrow & \{\text{predicate calculus and functional iteration}\} \\
 & (\underline{A}X :: [h(k^0 X) \Rrightarrow f\,X]) \quad .
\end{array}
$$

For the induction step we observe

$(\underline{A}\, X :: [h(k^i X) \Rightarrow f X])$

$\Rightarrow$ { restriction to X of the form $k\,Y$; functional iteration }

$(\underline{A}\, Y :: [h(k^{i+1} Y) \Rightarrow f(k\,Y)])$

$\Rightarrow$ { $f$ solves (4), hence $[f(k\,Y) \Rightarrow f\,Y]$ for any $Y$ }

$(\underline{A}\, Y :: [h(k^{i+1} Y] \Rightarrow f\,Y])$ .

(End of Proof.)

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