Solution to Exercise 1: Algebraic Curve, Surface Splines - I

CS384R, CAM 395T, BME 385J: Fall 2007

September 16, 2007

Question 1. The singularities of an algebraic plane curve $f(x, y) = 0$ are given by all solutions of $f(x, y) = f_x = f_y = 0$ where $f_x = \frac{\partial f(x,y)}{\partial x}$ and $f_y = \frac{\partial f(x,y)}{\partial y}$ are the partials of f with respect to x and y respectively.

(A) (a) $x^2 - y^7 = 0$

(b) $x^2 - y^3 - y^7 = 0$

(c) $x^3 + y^3 - 1 = 0$

(d) $2x^4 - 3x^2y + 2y^3 + y^4 = 0$

(B) Which of the curves in (A) are rational?

(C) Derive a parametric form for each of the curves, attempting to derive the simplest polynomial or rational form for F and G, whenever possible.

Sol. (a) $x^2 - y^7 = 0$, $2x = 0$, $7y^6 = 0$ i.e., (0, 0,1) is a singular point. Also, singularities at ∞ are easy to calculate via the projective curve. This can be done by the substituting $x = \frac{X}{W}, y = \frac{Y}{W}$ to map the (x, y) coordinate to (X, Y, W). This leads to, $x^2 - y^7 = 0 \Rightarrow X^2W^5 - Y^7 = 0 \Rightarrow w^5 - y^7 = 0$. Hence the singularity at infinity $(1, 0, 0)$ has multiplicity 5. Substituting $y = t^2$ in the equation yields $x = t^7$. Hence the given equation is rational.

(b) There is a singular point at (0,0,1) with multiplicity 2. There is a singularity at infinity (1,0,0) with multiplicity 5. Hence this curve is not rational.

(c) $x^3 + y^3 - 1 = 0$ has no singularities. Hence it has genus not equal to zero. Therefore, it is not rational. One may parametrize it as $x = \sqrt[3]{1 - t^3}$, $y = t$.

(d) The given equation has a singular point at $(0, 0)$ with multiplicity 2. To parametrize setting $y = xt$ one has $x^3(2x - 3t + 2t^3 + xt^4)$, thus $x = \frac{3t(1-t^2)}{2+t^4}$ and $y = \frac{3t^2(1-t^2)}{2+t^4}$. Therefore, it is rational.

Question 2. An algebraic hyperelement is an $(n - 1)$ dimensional set of points defined in n-dimensional space by a single multivariate polynomial equation $f(x_0, x_1, \ldots, x_{n-1}) = 0$ on n variables. For example, a plane algebraic curve given by $f(x_0, y_0) = 0$ is also a 2-dimensional hyperelement and an algebraic surface $f(x_0, y_0, z_0)$ is a 3-dimensional hyperelement. Give a constructive proof that any quadratic hyperelement $f(x_0, x_1, \ldots, x_n) = 0$ is rational i.e., derive a rational parameterization.
{Hint: Assume you have a point on the hyperelement and take a 1-dimensional family of lines through that point.}

Sol. An algebraic hyperelement $f(x_0, \ldots, x_{n-1}) = 0$ in an n-dimensional space of degree 2 will intersect a line at two points (including infinity). Without loss of generality consider the case of hyperelement passes through the origin. In general, this may be accomplished by the translation $(x_i = x'_i - \alpha_i)$, where α_i are the coordinates of any point on the hyperelement. The algebraic hyperelement may be expressed as:

$$f(x_0, \ldots, x_{n-1}) = \sum_{0 < i_0 + \ldots + i_{n-1} \leq 2} a_{i_0, \ldots, i_{n-1}} (x_0^{i_0} \cdots x_{n-1}^{i_{n-1}}) = 0$$

The parametric expression in t of a generic line in E^n is:

$$\begin{aligned} x_0 &= s_0 t \\ &\cdots \\ x_{n-1} &= s_{n-1} t \end{aligned}$$

Being the the equation system homogeneus one may set $s_{n-1} = 1$.

To intersect such a line with the hyperelement it suffices to substitute the parametric expression in the algebraic equation, $f(s_0 t, \ldots, s_{n-1} t)$:

$$\sum_{0 < i_0 + \ldots + i_{n-1} \leq 2} a_{i_0, \ldots, i_{n-1}} (s_{i_0}^{i_0} \cdots s_{i_{n-1}}^{i_{n-1}}) t^{(i_0 + \ldots + i_{n-1})} = 0$$

Due $a_{0 \ldots 0} = 0$ (passing through the origin) only terms in t^2 and t are present, thus, gathering:

$$t \left(t \left[\sum_{i_0 + \ldots + i_{n-1} = 2} a_{i_0, \ldots, i_{n-1}} (s_{i_0}^{i_0} \cdots s_{i_{n-1}}^{i_{n-1}}) \right] + \sum_{i_0 + \ldots + i_{n-1} = 1} a_{i_0, \ldots, i_{n-1}} (s_{i_0}^{i_0} \cdots s_{i_{n-1}}^{i_{n-1}}) \right) = 0$$

Solving for the other solution:

$$t = \frac{\sum_{i_0 + \ldots + i_{n-1} = 1} a_{i_0, \ldots, i_{n-1}} (s_{i_0}^{i_0} \cdots s_{i_{n-1}}^{i_{n-1}})}{\sum_{i_0 + \ldots + i_{n-1} = 2} a_{i_0, \ldots, i_{n-1}} (s_{i_0}^{i_0} \cdots s_{i_{n-1}}^{i_{n-1}})}$$

Substituting the above value of t in the parametric expression of the line one obtains a parametric expression in $s_1 \ldots s_{n-1}$ parameters, thus the parametric rational expression of the hypersurface:

$$\begin{aligned} x_0 &= s_0 \frac{\sum_{i_0 + \ldots + i_{n-1} = 1} a_{i_0, \ldots, i_{n-1}} (s_{i_0}^{i_0} \cdots s_{i_{n-1}}^{i_{n-1}})}{\sum_{i_0 + \ldots + i_{n-1} = 2} a_{i_0, \ldots, i_{n-1}} (s_{i_0}^{i_0} \cdots s_{i_{n-1}}^{i_{n-1}})} \\ &\cdots \\ x_{n-1} &= s_{n-1} \frac{\sum_{i_0 + \ldots + i_{n-1} = 1} a_{i_0, \ldots, i_{n-1}} (s_{i_0}^{i_0} \cdots s_{i_{n-1}}^{i_{n-1}})}{\sum_{i_0 + \ldots + i_{n-1} = 2} a_{i_0, \ldots, i_{n-1}} (s_{i_0}^{i_0} \cdots s_{i_{n-1}}^{i_{n-1}})} \end{aligned}$$

Question 3. An algebraic space curve segment C is a connected piece of an algebraic space curve. C can be represented by a pair of algebraic surfaces, alongwith a pair of vertices (a pair of points on the curve), denoted as the starting vertex v_1 and an ending vertex v_2, of the curve segment. Additionally, a vector t_1 is provided, which is tangent to the curve segment C at v_1, and specifies the points of the curve segment from v_1 to v_2. Compute the intersection of a given surface $S : x^2 + y^2 + z^2 - 1 = 0$ with a space curve segment C given by the pair of surfaces $(x^2 + y^2 - z = 0, x = 0)$, and a starting vertex $v_1 = (0, 2, 4)$ and an ending vertex $v_2 = (0, -2, 4)$. Furthermore, the tangent vector $t_1 = (0, 1, -4)$ at v_1 is also given.

Sol. Surface $S : x^2 + y^2 + z^2 - 1 = 0$ (sphere $(o = (0,0,0), r = 1)$).

Curve $C : (x^2 + y^2 - z = 0, x = 0)$ (plane $x = 0 \cap$ conicoid aloong z axis)

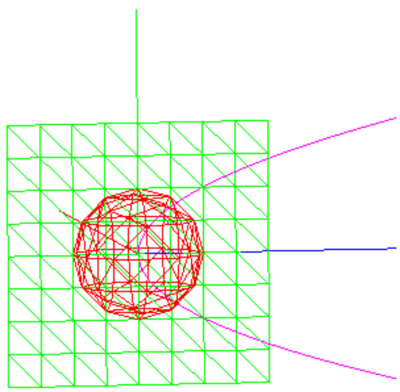
Parametrizing the plane $x = 0$ as $(x = 0, y = s, z = t)$ and substituting it in the conicoid: $(t = s^2)$ and in the plane back again:

$$x = 0$$
$$y = s$$
$$z = s^2$$

Points $v_1 = (0,2,4)$ and $v_2 = (0,-2,4)$ mean $s = 2$ and $s = -2$. Deriving the plane curve in s: $(\frac{dx}{ds} = 0, \frac{dy}{ds} = 1, \frac{dz}{ds} = 2s)$ the tangent $t_1 = (0,1,-4)$ is satisfied for $s = 2$ and $ds = -1$. This means the segment is defined by s is decreasing from v_1 to v_2 . Thus the segment is defined in $s \in [2 \ldots -2]$ passing through 0. One could reparametrize but it is not needed.

To intersect, one substitutes the parametric equation in the surface S having$s^2 + s^4 - 1 = 0$, solving one has $s^2 = t = \frac{-1 \pm \sqrt{5}}{2}$, excluding imaginary points $s = \pm\sqrt{\frac{-1+\sqrt{5}}{2}}$ (inside the range of s). The intesection points are:

$$\begin{aligned} x &= 0 \\ y &= \pm\sqrt{\tfrac{-1+\sqrt{5}}{2}} \approx \pm 0.7861 \\ z &= \tfrac{-1+\sqrt{5}}{2} \approx 0.6180 \end{aligned}$$



Question 4. A surface patch is a surface with boundary. It is also defined to be of finite area piece and therefore possesses boundary B_c which are cycles of affine curves segments lying on the surface. An algebraic surface patch P shall thus be represented by a single polynomial equation, and closed cycles of algebraic space curve segments lying on the surface. The algebraic surface patch points are defined to be the left of the algebraic curve segment cycles, when viewed from the space which contains the normal of the surface and when traversing the boundary cycles in counter-clockwise order. Compute the intersection of a given spherical surface patch P with (a) the plane $y = z$ and (b) the surface $y^2 + z^2 - 1 = 0$. The patch P is given by $x^2 + y^2 + z^2 - 1 = 0$, and boundary space curve segment cycle specified by an ordered cycle of vertices $(1,0,0), (0,-1,0), (-1,0,0), (0,1,0)$ lying on $x^2 + y^2 - 1 = 0$. Your answer should be represented as a collection of algebraic curve segments.

Sol. Surface patch SP is desribed by: $S : x^2 + y^2 + z^2 - 1 = 0$ (sphere $(o = (0,0,0), r = 1)$) delimited by $S_c : x^2 + y^2 - 1 = 0$ (cylinder on xy plane along z axis, centered) and ordered vertices $(1,0,0)$, $(0,-1,0)$, $(-1,0,0)$, $(0,1,0)$.

Intersect with S_a:y=z (plane bisecting y z axis) and $S_b : y^2 + z^2 - 1 = 0$ (cylinder on xz plane along y axis, centered).

First of all by elimination between S and S_c one obtains $z = 0$. This means any two of this plane, S_c or S describes the patch delimiting curve. It means also that the curve lies on the $z = 0$ plane. By the ordering of the vertex one obtains the patch is the half sphere pointing towards negative z axis.

$SP \cap S_a$: The curve equations describing the intersection is already fully described by S and S_a. However if one wants to compute another surface, containing the curve, which intersected with either S_a or S will describe the curve, one may use the $y = z$ equation to obtain either: $Res_{0,z}(S, S_a) : x^2 + 2y^2 - 1 = 0$ or $Res_{0,y}(S, S_a) : x^2 + 2z^2 - 1 = 0$. The may be usefull to find another vertex on curve segment for example when $x = 0$ one has $z = y = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = sin(\pi/4) = cos(\pi/4)$.
To compute the extreme vertex of the curve segment on must check the intersection $S \cap S_c \cap S_a$. Alternatevelly one may intersect $S_a, y = z$, and S. Obtaining $(x = \pm 1, y = 0, z = 0)$
To sum up $SP \cap S_a$ is the curve segment described by the intersections of any two of S_a, S, $Res_{0,z}(S, S_a)$,$Res_{0,y}(S, S_a)$. A convenctional choice is:

$$\begin{array}{ll} Res_{0,z}(S, S_a) : & x^2 + 2y^2 - 1 = 0 \\ Res_{1,z}(S, S_a) : & x - z = 0 \end{array}$$

Hence, the arc comprising $(1, 0, 0), (-1, 0, 0), (0, 0, 1)$ with $(1, 0, 0), (-1, 0, 0)$ as its endpoints is the required.

$SP \cap S_b$: Similar to the above, this is S_b itself.

Question 5. Consider an arrangement (collection) of spheres of varying radii in R^3 (atoms of a molecule). Each sphere in the arrangement is described by a 4-tuple (center-coordinates, radius). Given an arrangement of four spheres $\{(0, 0, 0, 1), (0, 0, 1, 0.75), (0, 1, 0, 0.75), (1, 0, 0, 0.25)\}$, compute a boundary patch representation of the union of the arrangement (spatial description of the molecule), enumerating the various patch descriptions. Do this in the most efficient manner possible and describe your computation method. { Hint: Pairwise spheres and triple-wise sphere intersections need to be computed. }

Sol. The equation of the spheres $S_1 = [o = (0, 0, 0), r = 1]$, $S_1 = [o = (0, 0, 1), r = 0.75]$,$S_1 = [o = (0, 1, 0), r = 0.75]$, $S_1 = [o = (1, 0, 0), r = 0.25]$ are:

$$\begin{array}{llll} S_1 : & x^2 + y^2 + z^2 - 1 & = x^2 + y^2 + z^2 - 1 & = 0 \\ S_2 : & x^2 + y^2 + (z-1)^2 - 9/16 & = x^2 + y^2 + z^2 - 2z + 7/16 & = 0 \\ S_3 : & x^2 + (y-1)^2 + z^2 - 9/16 & = x^2 + y^2 + z^2 - 2y + 7/16 & = 0 \\ S_4 : & (x-1)^2 + y^2 + z^2 - 1/4 & = x^2 + y^2 + z^2 - 2x + 3/4 & = 0 \end{array}$$

$C_{12} = S_1 \cap S_2$:

$$\begin{array}{cccccccccc} x^2 & + & y^2 & + & z^2 & & & - & 1 & = & 0 \\ x^2 & + & y^2 & + & z^2 & - & 2z & + & 7/16 & = & 0 \\ \hline & & & & & & 2z & - & 23/16 & & 0 \end{array}$$

Thus $z = 23/32$, substituting in $S_0 : x^2 + y^2 + 529/1024 - 1 = x^2 + y^2 - 495/1024 = 0$

$C_{13} = S_1 \cap S_3$:

$$\begin{array}{cccccccccc} x^2 & + & y^2 & + & z^2 & & & - & 1 & = & 0 \\ x^2 & + & y^2 & + & z^2 & - & 2y & + & 7/16 & = & 0 \\ \hline & & & & & & 2y & - & 23/16 & = & 0 \end{array}$$

Thus $y = 23/32$, substituting in $S_1 : x^2 + z^2 - 495/1024 = 0$

$C_{14} = S_1 \cap S_4$:

$$\begin{array}{ccccccccccc} x^2 & + & y^2 & + & z^2 & & & - & 1 & = & 0 \\ x^2 & + & y^2 & + & z^2 & - & 2x & + & 3/4 & = & 0 \\ \hline & & & & & & 2x & - & 7/4 & = & 0 \end{array}$$

Thus $x = 7/8$, substituting in $S_1 : 49/64 + y^2 + z^2 - 1 = y^2 + z^2 - 15/6 = 0$

$C_{23} = S_2 \cap S_3$:

$$\begin{array}{cccccccccc} x^2 & + & y^2 & + & z^2 & - & 2z & + & 7/16 & = & 0 \\ x^2 & + & y^2 & + & z^2 & - & 2y & + & 7/16 & = & 0 \\ \hline & & & & & & 2y - 2z & - & 0 & = & 0 \end{array}$$

Thus $z = y$, Substituting in either S_2 or S_3: $2y^2 - 2y + x^2 + 7/16$

$C_{24} = S_2 \cap S_4$:

$$\begin{array}{ccccccccccc} x^2 & + & y^2 & + & z^2 & - & 2z & + & 7/16 & = & 0 \\ x^2 & + & y^2 & + & z^2 & - & 2x & + & 3/4 & = & 0 \\ \hline & & & & & & 2x - 2z & - & \frac{5}{16} & = & 0 \end{array}$$

Thus $x - z - \frac{5}{32} = 0$ $(z = x - 5/32)$, Substituting in S_4: $2x^2 + y^2 - 37x/16 + 793/1024$

Writing the poly in x and checking the discriminant one has $(-\frac{37}{16})^2 - 8(y^2 + 793/1024) \simeq -8y^2 + 5.347 - 6.195$ and is always negative for any y. The intersection is empty.

$C_{34} = S_3 \cap S_4$:

$$\begin{array}{ccccccccccc} x^2 & + & y^2 & + & z^2 & - & 2y & + & 7/16 & = & 0 \\ x^2 & + & y^2 & + & z^2 & - & 2x & + & 3/4 & = & 0 \\ \hline & & & & & & 2x - 2y & - & \frac{5}{16} & = & 0 \end{array}$$

Thus $x - y + \frac{5}{32} = 0$ $(y = x - 5/32)$, Substituting in S_4: $2x^2 + z^2 - 37x/16 + 793/1024$

Similarly the discrimintant is always negative and the intersection is empty.

$\{P_{123}\} = C_{12} \cap C_{23} = S_1 \cap (y = 23/32) \cap (z = 23/32) = \phi$: Substituting in $S_1 : X^2 + 529/1024 + 529/1024 - 1$ one has only complex solutions

$\{P_{234}\} = C_{23} \cap C_{34} = \phi \ \ (C_{34} = \phi)$

$\{P_{134}\} = C_{13} \cap C_{34} = \phi \ \ (C_{34} = \phi)$

$\{P_{124}\} = C_{12} \cap C_{24} = \phi \ \ (C_{24} = \phi)$

To sum up, the requisite surface comprising the patches formed by every point on S_2 which satisfies $(2z \geq 23/16)$, every point on S_3 which satisfies $(2y \geq 23/16)$, every point on S_4 which satisifies $(2x \geq 7/4)$, every point on S_1 which satisfies $(2z \leq 23/16)$ and $(2y \leq 23/16)$ and $(2x \leq 7/4)$.