Solutions to Exercise 2: Algebraic Curve, Surface Splines - II

CS384R, CAM 395T, BME 385J: Fall 2007

September 21, 2007

Question 1. Consider a simple polygon with $n \geq 3$ vertices. Describe a cubic A-spline construction scheme to smooth the polygon such that,

(a) vertices of Q are C^1-interpolated (interpolatory spline)

(b) vertices of Q are approximately interpolated to a user specified $\epsilon > 0$ (approximatory spline)

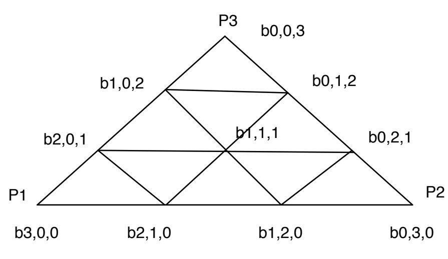
Sol: To define an A-spline segment one may use segments of the zero of a degree $d > 0$, Bernstein Bezier bivariate polynomial defined on a triangle :

$$f = \sum_{|i,j,k|=d} b_{i,j,k} \frac{n!}{i!j!k!} \alpha_1^i \alpha_2^j \alpha_3^k = 0$$

Where the Barycentric coordinates $\alpha_{\_i}$ are related to the Cartesian coordinate system by the vertices of the triangle:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} p_{1,x} & p_{2,x} & p_{2,x} \\ p_{1,y} & p_{2,y} & p_{2,y} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

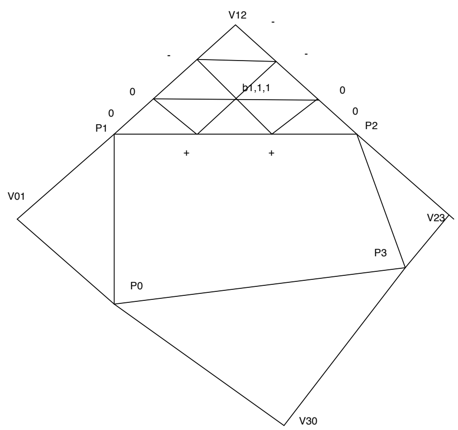
For the A-spline of cubic degree, shown in the figure below, one sets the coefficients $b_{3,0,0} = 0$ and $b_{0,3,0} = 0$ then it is easy to see that the curve interpolates p_1 and p_2 of the triangle. Hence one may consider p_1 and p_2 as the join points in constructing longer chains of segments and thereby achieve a C^0 continuous A-spline. To facilitate C^1 continuity at the curve segment end points (join points), one may additionally set $b_{2,0,1} = b_{0,2,1} = 0$ so that the A-spline curve segment inside the triangle, becomes tangent to $p_3 \vec{-} p_1$ at p_1 and to $p_3 \vec{-} p_2$ at p_2. We shall use these below to smooth polygons.



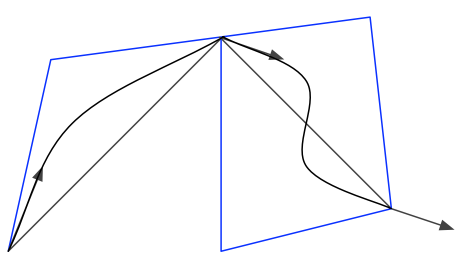
To achieve smooth (non-singular) and simply connected A-splines, one imposes a single sign change condition or each segment coefficients from the apex p_3 to the triangle base p_1, p_2. One way to do this is to set for each triangle, its $b_{2,1,0}$ and $b_{1,2,0}$ coefficients to be positive and its $b_{0,0,3}$, $b_{1,0,2}$, and $b_{0,1,2}$ to be negative.

Note, that the general conditions for C^0 or G^0 and C^1 or G^1 continuity between a pair of algebraic curve segments $f(x,y)=0$ and $g(x,y)=0$, at say their join point p, are the linear system of equations (in terms of the coefficients of polynomials f and g), and given by $f=g$ and $Tangent(f)=f_x y - f_y x = Tangent(g) = g_x y - g_y x$ evaluated at p.

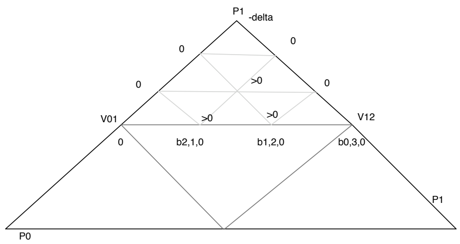
One way to construct a smoothed polygon, using the aforementioned endpoint tangent A-splines, and to achieve C^1 interpolation of the given polygon's vertices, is as follows. Assign a single tangent to each vertex of the polygon which is not coincident with either of the polygon's edges at that vertex. Extend the tangents from each vertex in both directions till they intersect with the extensions of the tangents of its two adjacent vertices thus constructing a triangle on each edge of the polygon. Assign $b_{3,0,0}=0$, $b_{0,3,0}=0$, $b_{2,0,1}=0$, and $b_{0,2,1}=0$ and $b_{2,1,0}$, $b_{1,2,0}$ equal to any positive real number, and $b_{0,0,3}$, $b_{1,0,2}$, $b_{0,1,2}$ to be negative. If the triangles overlap, the curve segment may overlap too, so reposition vertices of the newly constructed triangles such that triangles overlap is eliminated. This simple scheme, as outlined, only handles convex polygons, as shown below.



To achieve non–convex polygon smoothing, with vertex C^1 interpolation, one chooses a new auxiliary vertex on each of the two reflex edges which are incident to a reflex vertex (inner dihedral angle between reflex edges incident at the reflext vertex is greater than 180 degrees). Through each auxiliary vertex take a line making a positive angle with the reflex edge, and intersecting the two extended tangents of the two vertices defining the reflex edge. This then constructs two triangles per reflex edge, one on either side of the line. The assignment of coefficients on these triangles proceeds as before, to yield a C^1 smooth vertex interpolatory A-spline of any simple polygon.



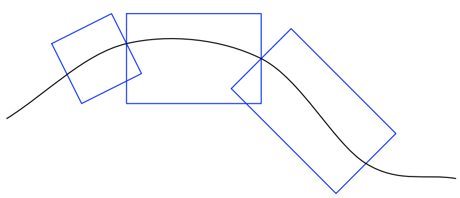
To generate a C^1 A-spline approximation of the vertices of any simple polygon (even non-convex), one may use the following simple scheme. Choose an auxiliary vertex (typically midway) on each polygon edge. Next for each adjacent polygon edge pair, connect the pair of auxiliary vertices . This forms a collection of triangles, where each triangle has 1 polygon vertex as its apex, and a pair of connected auxiliary vertices, as its base. Then as before, assign coefficients to the A-spline segments in each triangle with the same rule, $b_{3,0,0} = 0$, $b_{0,3,0} = 0$, $b_{2,0,1} = 0$, and $b_{0,2,1} = 0$ and $b_{2,1,0}$, $b_{1,2,0}$ equal to any positive real number, and $b_{1,0,2}$, $b_{0,1,2}$ to be any negative number. Finally, for any given $\epsilon > 0$ set $b_{003} = -\delta$ such the curve passes at the distance ϵ from the vertex of the polygon. δ is usually taken to be $0.5 * \epsilon$. If $b_{003} = 0$ the curve would factor into two lines.



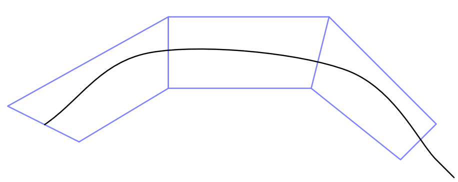
One may also define C^1 A-splines using the barycentric coordinates on a rectangle (p_0, p_1, p_2, p_3).

$$f(\alpha_1, \alpha_2) = \sum_{i=0}^{n} \sum_{j=0}^{m} b_{ij} \begin{pmatrix} n \\ i \end{pmatrix} \alpha_1^i (1-\alpha_1)^{2-i} \begin{pmatrix} m \\ j \end{pmatrix} \alpha_2^j (1-\alpha_2)^{2-j} = 0$$

$$\begin{bmatrix} r \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} (p_{1,x} - p_{2,x}) & (p_{1,x} - p_{3,x}) & p_{1,x} \\ (p_{1,z} - p_{2,y}) & (p_{1,y} - p_{3,y}) & p_{1,y} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 1 \end{bmatrix}$$



Lastly one may construct A-spline segments within skew quadrilateral domains, with the oposing sides normal to the curve .



Question 2. Consider a spatial polygonal chain $Q^s \subseteq R^3$ with starting vertex $\overrightarrow{P_0}$ and ending vertex $\overrightarrow{P_n}$, $n \geq 3$. Construct a cubic A-spline, D, to C^1-interpolate the vertices of Q^s. Quantify the degrees of freedom of D that are still available to modify the spline without changing the topology of the polygonal chain Q^s.

Sol: Let $p_0, p_1, \ldots, p_n$ be the sequence of vertices along the given polygonal chain (w.l.o.g. assume that $n = 3m - 1$, for an integer m). Consider three successive vertices p_i, p_{i+1}, p_{i+2}, for $i = 3m'$ with m' an integer. Let R be the rectangle with smallest area such that it contains the triangle formed with the points p_i, p_{i+1}, p_{i+2}. Since we require a cubic A-spline, there are 4 control points (including the vertices) equi-spaced along each side of the rectangle R. Let these control points , while traversing R in clockwise direction, be listed as $c_1 = p_i, c_2, c_3, c_4 = p_{i+2}, c_5, \ldots, c_{12}$. Suppose the point p_{i+1} incidents in between the control points c_8 and c_9. We chose weights at these control points s.t. $w_{p_i} = w_{p_{i+2}} = 0$; w_{c_2} and w_{c_3} as $-ve$; the weights of the remaining control points are chosen as +ve.

Let $T_j = \{p_i, p_{i+1}, p_{i+2}\}, T_{j+1} = \{p_{i+2}, p_{i+3}, p_{i+4}\}$ be two consecutive tensor patches constrcuted as described above. The tangent at p_{i+2} for the A-spline generated due to T_j be t_j. We add a constraint s.t. t_j is the tangent to the A-spline generated in T_{j+1}. This is accomplished by choosing the weights of the control points in T_{j+1}.

Since two vectors (one tangent, one binormal) constrain the plane in which an A-spline traverses through, defining only one of them leaves three degress of freedom at point p_{i+2}. When we consider the composite A-spline this leaves $3(\frac{n}{3})$ i.e., n degrees of freedom. What about the degrees of freedom with a given A-spline? (we are fixing only eight control points weights among the twelve that define the cubic A-spline)

Question 3. Consider a C^1-interpolatory quadratic A-spline, D, defined in the $x = 0$ plane (i.e., Y-Z plane) with none of the vertices $\overrightarrow{P_0}, \overrightarrow{P_1}, \ldots, \overrightarrow{P_n}$ incident on the z-axis.

(a) Describe a square pyramidal A-patch data structure that represents the spline surface of revolution generated when D is revolved about the Z-axis.

(b) What is the degree of the spline surface?

(c) What property of the A-spline would yield a lower degree spline surface of revolution?

(d) Convert the square-pyramidal representation to a tetrahedral A-patch representation.

Sol: The surface of revolution of an algebraic plane curve $f(y, z) = 0$ about the Z axis, can be described by the implicit equation algebraic given by $f(r, z) = f(\sqrt{x^2 + y^2}, z) = 0$, since the circle of revolution is $x^2 + y^2 = r^2$, and $r = y$, when $x = 0$ and r doesn't change. The single square root can be easily removed by separating all terms involving the square-root on one-side of the equation, and the rest of the terms on the other, and then powering both sides without changing the resulting zeroset of the surface equation. If the plane curve is of degree d, and has y-terms with only even exponents ,y^{2i}, $0 \geq i \leq d$, then the sqare root is already eliminated, and the resulting degree of the surface of revolution is not doubled.

The spline curve in the Y-Z plane, may be constructed, using any kind of A-spline (see question 1). for example, let the spline be described by a rectangular quadratic A-spline (cfr question 1) the equation of each segment will be:

$$f(\alpha_1, \alpha_2) = \sum_{i=0}^{2} \sum_{j=0}^{2} b_{ij} \begin{pmatrix} n \\ i \end{pmatrix} \alpha_1^i (1 - \alpha_1)^{2-i} \begin{pmatrix} n \\ j \end{pmatrix} \alpha_2^j (1 - \alpha_2)^{2-j} = 0$$

Where the $\alpha_{\_}$ are the local coordinates in the scaffolding rectangle (p_0, p_1, p_2, p_3) of the segment, related to the (r, z) coordinates by:

$$\begin{bmatrix} r \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} (p_{1,r} - p_{2,r}) & (p_{1,r} - p_{3,,r}) & p_{1,r} \\ (p_{1,z} - p_{2,z}) & (p_{1,z} - p_{3,z}) & p_{1,z} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 1 \end{bmatrix}$$

Inverting said relation one has:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 1 \end{bmatrix} = \begin{bmatrix} (p_{1,r} - p_{2,r}) & (p_{1,r} - p_{3,,r}) & p_{1,r} \\ (p_{1,z} - p_{2,z}) & (p_{1,z} - p_{3,z}) & p_{1,z} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r \\ z \\ 1 \end{bmatrix}$$

Substituting the $\alpha\_ = \alpha(r, z)$ in f one has a form $f(r, z) = 0$, thus effectivelly transforming the equation in power basis. Alternatevelly one could have set up the transformation matrix. Applying the rotation one has the surface $f(r, y, z) = f(x, y, z) = 0$ in the current coordinate system.

At this point one has to define a new a-patch scaffold and transform the surface equation into the new scaffold coordinate system. The new scaffold *must* have the two sides parallel to the xy plane. Otherwise the border may not be a planar curve, furthermore, the border curve segments will be circumpherences. (single valued and symetry)

Say, v_0 and v_1 be the original extreme points of the curve segment. Let a be the maximum r, and $b_1$and b_2 the ordered z coordinates of v_1 and v_2. Build the cuboid at $(0, b_1)$ with height $b_2 - b_1$ and both sides equal to a. This is a rectangular scaffold for the a-patch. To build the square pyramidal a-patch one may split the cuboid in four pyramids at the barycentric point $(r/2\ , r/2.,\ b_1 + (b_2 - b_1)/2)$.

At this point one has to convert back into tensorial form for the rectangular patches:

$$f(\alpha_1, \alpha_2) = \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{k=0}^{2} b_{ij} \left(\begin{array}{c} n \\ i \end{array} \right) \alpha_1^i (1-\alpha_1)^{2-i} \left(\begin{array}{c} n \\ j \end{array} \right) \alpha_2^j (1-\alpha_2)^{2-j} \left(\begin{array}{c} n \\ k \end{array} \right) \alpha_3^k (1-\alpha_3)^{2-k} = 0$$

From the coordinate conversion matrix:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 1 & 1 & b_2 - b_1 & b_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 1 \end{bmatrix}$$

Or to the mixed tensor barycentric form for the pyramidal patch:

$$f(\alpha_1, \alpha_2) = \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{k=0}^{2} b_{ij} \left(\begin{array}{c} n \\ i \end{array} \right) \alpha_1^i (1-\alpha_1)^{2-i} \left(\begin{array}{c} n \\ j \end{array} \right) \alpha_2^j (1-\alpha_2)^{2-j} \left(\begin{array}{c} n \\ k \end{array} \right) \alpha_3^k = 0$$

Where coordinates conversion depend on which pyramid, for example is the bottom one:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} r & 0 & r/2 & 0 \\ 0 & r & r/2 & 0 \\ 1 & 1 & (b_2 - b_1)/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 1 \end{bmatrix}$$

In any case the resulting surface will be biquadratic.

Question 4. Consider two circles in R^3, of radii 1 and 2, lying on the $x = 1$ and $y = 4$ planes, and with their centers on the x and y axis respectively.

(a) Compute A-spline representation of each circle.

(b) Compute a joining surface that interpolates the circles and contains the origin. Give your answer as an A-patch representation.

(c) Is your solution the lowest degree algebraic surface and with the fewest number of A-patches ?

Sol: (a) We construct a scaffold around the circle in the following way. Consider the unit circle C_1 whose centre is located at $x = 1$. We choose the bivariate triangle scaffold TS (supplementary lecture note 2 slide 4) around each quarter circle defined w.r.t. y and z axes s.t. the $\angle a_{020}a_{200}a_{002} = \frac{\pi}{2}$ and the scaffold is coplanar with C_1. As circle is quadratic, there are 3 (including the vertices) equi-spaced control points along each side of TS.

We associate weights to points $a_{020}, a_{011}, a_{002}, a_{101}, a_{200}, a_{110}$. Since C_1 need to interpolate a_{020} and a_{002}, we set $w_{020} = w_{002} = 0$. Since the line segments $a_{020}a_{110}$ and $a_{002}a_{101}$ are tangents to C_1, we set $w_{110} = w_{101} = 0$. Consider the line segment $a_{200}a_{011}$ and let the point of intersection of the circle embedded in the triangle with this line segment be p. From the basic geometry, $|\frac{pa_{200}}{pa_{011}}| = \frac{w_{011}}{w_{200}} \Rightarrow |\frac{\sqrt{2}-1}{1-\frac{1}{\sqrt{2}}}| = \frac{w_{011}}{w_{200}} \Rightarrow |\frac{1}{\sqrt{2}}| = \frac{w_{011}}{w_{200}}$. We choose $w_{011} = -1$ and $w_{200} = \sqrt{2}$ to satisfy this equality. This scaffold is guaranteed to yield the one quarter of the circle. The scaffolds for circle quarters in the other quadrants can be constructed symmetrically. For one of these scaffolds, say $a'_{020}a'_{002}a'_{200}$, distinct from TS, since $\angle a_{020}a_{200}a_{002} = \angle a'_{020}a'_{200}a'_{002} = \frac{\pi}{2}$, $a'_{200}a'_{020}$ is collinear with either $a_{200}a_{002}$ or $a_{020}a_{200}$. This helps in keeping the C_1 continuity of the circle segments at the joining points of the generated A-splines.

The A-spline representation of the circle C_2 located in $y = 4$ is symmetric to above except that the scffolds are constructed in the xz-plane.

(b) The circle C_1 can be represented as an intersection of two implicit surface equations, $f_0 : y^2 + z^2 - 1 = 0$ and $g_0 : x - 1 = 0$. The circle C_2 can be represented as an intersection of two implicit surface equations, $f_1 : x^2 + z^2 - 4 = 0$ and $g_1 : y - 4 = 0$. Then the lofted surface which interpolates (i.e., with C^0 continutity) the circles C_1 and C_2 is given by, $\alpha_0 f_0 + \beta_0 g_0^{0+1} = \alpha_1 f_1 + \beta_1 g_1^{0+1} \Rightarrow \alpha_0(y^2 + z^2 - 1) + \beta_0(x - 1) = \alpha_1(x^2 + z^2 - 4) + \beta_1(y - 4)$, where $\alpha_0, \beta_0, \alpha_1, \beta_1$ are chosen to be constants to have the lowest degree algebraic surface. This yields the surface S, $\alpha_1 x^2 - \alpha_0 y^2 + (\alpha_1 - \alpha_0)z^2 + \beta_1 y - \beta_0 x + (-4\alpha_1 - 4\beta_1 + \alpha_0 + \beta_0) = 0$. Take any four points on the circles C_1 and C_2 to find these constants, ex. $(2, 4, 0), (0, 4, 2), (1, 1, 0), (1, 0, 1)$. The additional constraint is given by that the origin $(0, 0, 0)$ need to be interior to the surface. This can be imposed by $\frac{\partial S}{\partial y} < 0$ at $(2, 4, 0)$.

We define A-patches in tensor domain. Consider the smallest hexahedron H enclosing the surface S. We partition H with a set R of hexahedrons s.t. any line segment joining vertices of any chosen hexahedron in R does not intersect S more than once, and, $|R|$ is the smallest among all possible such sets. Since S is quadratic, each side of any hexahedron $r \in R$ consists of three control points (including the vertices). Similar to part (a), we associate weights to these control points to construct A-patch within each such r. To maintain C^2 continuity, between adjacent A-patches, we impose additional constraints.

(c) As mentioned in lecture note 2 slide 55, the equation representing the family of surfaces which G^k interpolate the curves C_1 and C_2 is $\alpha_0 f_0 + \beta_0 g_0^{k+1} = \alpha_1 f_1 + \beta_1 g_1^{k+1}$. Since we were asked to interpolate C_1 and C_2, we have chosen $k = 0$, which in turn yielded the lowest degree algebraic surface. Since S is of lowest degree and $|R|$ is of smallest size, our solution is with the fewest number of A-patches.

Question 5. The topology of an A-patch is the local topology of the level set of the algebraic function within its bounding linear finite element (simple polyhedron), and related to the critical points of the algebraic function within the finite element. Describe the possible topologies of (a) quadratic A-patch in a tetrahedron (b) tri-linear A-patch in a cube (c) combination [linear,quadratic] A-patch in a triangular prism

Sol: See class notes.