Solutions to Exercise 3: Algebraic Curve, Surface Splines - III

CS 384R, CAM 395T, BME 385J: Fall 2007

October 02, 2007

Question 1. Consider the normal r-offset surfaces Q_{outer} and Q_{inner} of an algebraic surface patch P inside a tetrahedron, where Q_{outer} is the offset in the positive surface normal direction and Q_{inner} is the offset in the negative surface normal direction by r. If patch P is defined by a quadratic trivariate polynomial equation, give the equation of the Q_{outer} and Q_{inner} surfaces and the patch boundaries within a r-offset (or r-scaled) tetrahedron.

Sol. Say patch P be defined by

$$P : F(x, y, z) = 0 \tag{1}$$

then the normal of P at point (x, y, z) is

$$\vec{n} \quad = (n_x, n_y, n_z)^T \quad = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})^T$$

Denote

$$| \vec{n} |= \sqrt{n_x^2 + n_y^2 + n_z^2}.$$

Let (x', y', z') be the point on Q_{outer}, then we have

$$\begin{cases} x' = x + r\dfrac{n_x}{| \vec{n} |} \\ y' = y + r\dfrac{n_y}{| \vec{n} |} \\ z' = z + r\dfrac{n_z}{| \vec{n} |} \end{cases} \tag{2}$$

Eliminate x, y, z from (**??**) and (**??**) to get an equation about x', y', z'. It is the equation of Q_{outer}.

Let $(\tilde{x}, \tilde{y}, \tilde{z})$ be the point on Q_{inner}, then

$$\begin{cases} \tilde{x} = x - r\dfrac{n_x}{| \vec{n} |} \\ \tilde{y} = y - r\dfrac{n_y}{| \vec{n} |} \\ \tilde{z} = z - r\dfrac{n_z}{| \vec{n} |} \end{cases} \tag{3}$$

Eliminate x, y, z from (**??**) and (**??**) to get an equation about $\tilde{x}, \tilde{y}, \tilde{z}$. It is the equation of Q_{inner}.

Here we give an example to show how to get the equation about x', y', z'.

Let $F(x, y, z) = x^2 + y^2 + z^2 - 1$, then

$$\vec{n} \quad = (n_x, n_y, n_z)^T \quad = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})^T \quad = (2x, 2y, 2z)^T$$

$$| \vec{n} |= \sqrt{n_x^2+n_y^2+n_z^2} = 2\sqrt{x^2+y^2+z^2} = 2.$$

Then

$$\begin{cases} x' = x + r\dfrac{n_x}{| \vec{n} |} = x + r\dfrac{2x}{2} = (1+r)x \\ y' = y + r\dfrac{n_y}{| \vec{n} |} = y + r\dfrac{2y}{2} = (1+r)y \\ z' = z + r\dfrac{n_z}{| \vec{n} |} = z + r\dfrac{2z}{2} = (1+r)z \end{cases} \tag{4}$$

So

$$x = x'/(1+r), y = y'/(1+r), z = z'/(1+r).$$

Substitute is to $F(x, y, z) = x^2 + y^2 + z^1 - 1 = 0$, we get

$$F'(x', y', z') = (x'^2 + y'2 + z'^2)/(1 + r)^2 - 1 = 0$$

which is the equation of Q_{outer}.

Similarly we can get the equation of Q_{inner}:

$$\tilde{F}(\tilde{x}, \tilde{y}, \tilde{z}) = (\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)/(1 - r)^2 - 1 = 0$$

Let the patch P be defined in the tetrahedron $\mathcal{T} : (p_1, p_2, p_3, p_4)$, where $p_1 = (x_1, y_1, z_1), p_2 = (x_2, y_2, z_2), p_3 = (x_3, y_3, z_3), p_4 = (x_4, y_4, z_4)$. Notice that p_1, p_2, p_3 are on the surface Q_{outer}, then

$$.x_1^2 + y_1^2 + z_1^2 = (1 + r)^2, x_2^2 + y_2^2 + z_2^2 = (1 + r)^2, x_3^2 + y_3^2 + z_3^2 = (1 + r)^2.$$

The relationship of the barycentric coordinates $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and (x, y, z) is:

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = M \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

Substitute the above equation into $F'(x', y', z') = 0$, we get the equation about the barycentric coordinates

$$\begin{aligned} G'(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \quad = \quad & \left({x_1}^2 + {z_1}^2 + {y_1}^2\right) {\alpha_1}^2 + (2\, y_2\, y_1 + 2\, x_2\, x_1 + 2\, z_2\, z_1)\, \alpha_1\, \alpha_2 \\ & + (2\, (y_3 + y_4)\, y_1 + 2\, (x_3 + x_4)\, x_1 + 2\, (z_3 + z_4)\, z_1)\, \alpha_3\, \alpha_1 + \left({x_2}^2 + {z_2}^2 + {y_2}^2\right) {\alpha_2}^2 \\ & + (2\, (x_3 + x_4)\, x_2 + 2\, (z_3 + z_4)\, z_2 + 2\, (y_3 + y_4)\, y_2)\, \alpha_3\, \alpha_2 \\ & + \left((x_3 + x_4)^2 + (z_3 + z_4)^2 + (y_3 + y_4)^2\right) {\alpha_3}^2 - (1 + r)^2 \\ = \quad & (1 + r)^2 {\alpha_1}^2 + (2\, y_2\, y_1 + 2\, x_2\, x_1 + 2\, z_2\, z_1)\, \alpha_1\, \alpha_2 \\ & + (2\, (y_3 + y_4)\, y_1 + 2\, (x_3 + x_4)\, x_1 + 2\, (z_3 + z_4)\, z_1)\, \alpha_3\, \alpha_1 + (1 + r)^2 {\alpha_2}^2 \\ & + (2\, (x_3 + x_4)\, x_2 + 2\, (z_3 + z_4)\, z_2 + 2\, (y_3 + y_4)\, y_2)\, \alpha_3\, \alpha_2 \\ & + \left((1 + r)^2 + 2x_3x_4 + 2z_3z_4 + 2y_3y_4 + x_4^2 + y_4^2 + z_4^2\right) {\alpha_3}^2 - (1 + r)^2 \end{aligned}$$

In many applications not only the surface equation must be offset but also the boundary. When offsetting surface patches that are not C^1 continuous the newly created surfaces will join the offset surfaces. The offset of a curve is obtained by moving a sphere on the curve. Equivalently one moves a circumference on a plane perpendicular to the tangent of the curve. A curve in space is defined as the intersection of two surfaces. The boundary curves in a-patches are the intersection of the surface equation and the plane containing the face of the scaffolding tetrahedron.

Surface: $f(x, y, z) = 0$

Delimiting plane: $P : ax + by + cz + d = 0$

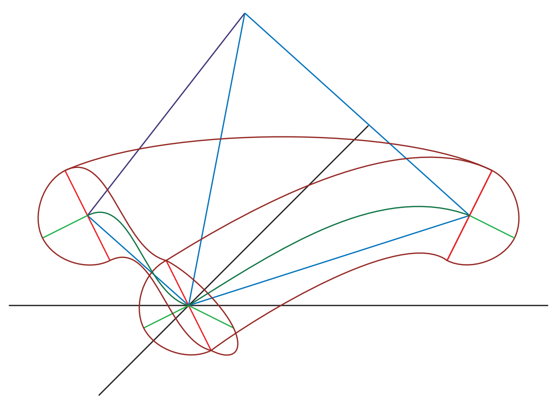
Normal to surfaces: $\overrightarrow{n_f} := \nabla f = (f_x, f_y, f_z)^T$ and $\hat{n_f} = \frac{\nabla f}{|\nabla f|}$ (in figure shown in red at curve segment delimiting points)

Normal to plane: $\overrightarrow{n_P} = (a, b, c)$ and $\hat{n}_P = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$.

Tangent to boundary curve: $\hat{t} = \hat{n_f} \times \hat{n_p}$

For any α, β one has $(\alpha\hat{n_f} + \beta\hat{n_P}) \cdot \hat{t} = 0$. The circunference will lay on the plane spanned by these two vectors.

In particular $\hat{b} = \hat{n_f} \times \hat{t}$ (shown in green at delimiting point)

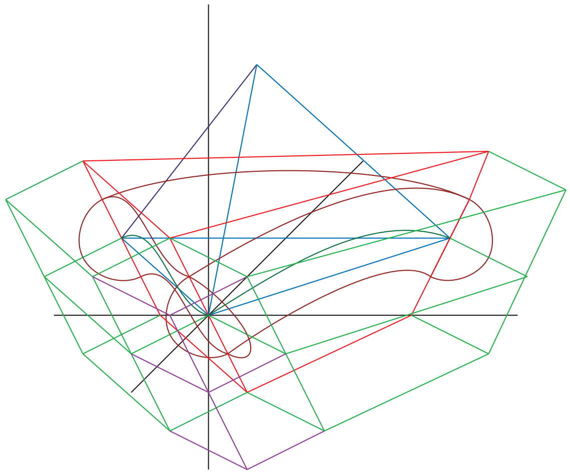


To build the scaffold for the offset surface and the offset of the delimiting curve and points for simplicity will focus on the original a-patch interpolating three points on the tetrahedron. The most natural scaffold is a prismatic scaffold, then one may add a baricentric point in each prism to divide it in 6 tetrahedrons.

The main surface positive offset surface Q_{outer} scaffold is build by taking the normal versor $\hat{n}$ at the boundary points and multiply it by two times the offset radius $2r$. Join the other extreme of $2r\hat{n}$ obtaining a prism skew in one direction. The offset surface will pass at the boundary edge exactly at $r\hat{n}$. Similarly one builds the prism for Q_{inner} by inverting the normals (red in figure).

The scaffold offset of the boundary curve is obtained by taking the binormal versor $\hat{b} = \hat{n_f} \times \hat{t}$, multiplying it by $2r$ and applying at the original boundary point and the boundary point at the end of $2r\hat{n}$. (green in figure)

The scaffold for the offset is obtained by building an hexahedron on the extended points on the bi-normals. (purple in figure)



Question 2. Given a union M of n spheres (simple geometric model of a molecule), give an efficient algorithm to generate the r-offset M_r^+ and M_r^- models of M where again M_r^+ is the outer offset and M_r^- the inner offset. What is the relationship of the inner r-offset $(M_r^+)_r^-$ of M_r^+ with M? Provide an algorithm to generate a model of $(M_r^+)_r^-$.

Sol. A surface patch decomposition M of an union of n spheres centered at different points $(p_0 = (x_0, y_0, x_0), p_1 = (x_1, y_1, x_1), \dots p_{n-1} = (x_{n-1}, y_{n-1}, x_{n-1}))$ and with different radii $(r_0, r_1, \dots, r_{n-1})$ is composed of patches whose components are:

1. Spherical surfaces: $S_i : (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 - r_i^2 = 0$
2. Boundary curves: these may be defined directly as intersection of the spheres $C_{ij} : S_i \cap S_j$ if not empty, but may simplified to a plane P_{ij} and a circunference c_{ij} on that plane.
   - Without loss of generality consider the coordinate system centered in p_i center of the first sphere and the x axis pointing to p_j center of the second sphere. Let d_{ij} be the distance between the two centers. vector.
   - One has:
   $$\begin{array}{c} x^2 + y^2 + z^2 - r_i^2 = 0 \\ (x - d_{ij})^2 + y^2 + z^2 - r_j^2 = 0 \\ \hline -2d_{ij}x + d_{ij}^2 + (r_j^2 - r_i^2) = 0 \end{array}$$
   - Thus the plane (in the local system) is
   $$x = h'_{ij} = \frac{d_{ij}^2 + (r_j^2 - r_i^2)}{2d_{ij}}$$
   - Substituting in the sphere equation one has the equation of the circunference and its radius:
   $$y^2 + z^2 - r_ij^2 = 0$$
   $$r'^2_{ij} = r_i^2 - h'^2_{ij} = r_i^2 - \left(\tfrac{d_{ij}^2 + r_j^2 - r_i^2}{2d_{ij}}\right)^2$$
3. Intersection points of three spheres $p_{ijk} = S_i \cap S_j \cap S_k$. One may more easy intersect the planes of the boundary curves: $p_{ijk} = P_i \cap P_j \cap P_k$.

Once the intersection planes, circumferences and points one may assemble the surface patch for M (See Figure **??**).

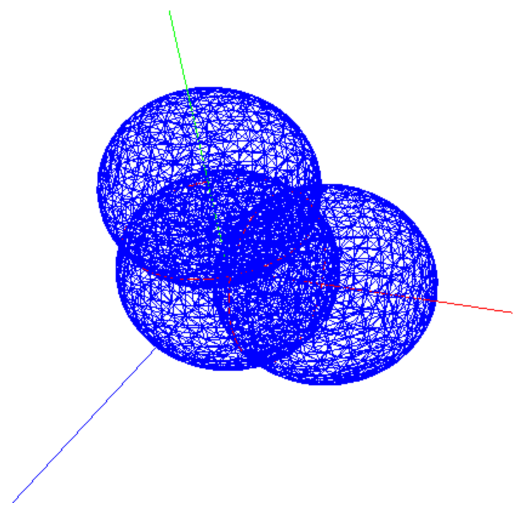


Figure 1: Three spheres and intersections

To calculate the an offset one has to r offset each component of the surface patch in the space [**?**][**?**]:

1. The r offset of spherical surface is very simple. Being the normal to a sphere always pointing in the direction of the radius, the offset is just the original surface equation with the the added radius:

$$(S_r^+)_i : (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 - (r + r_i)^2 = 0$$

2. The r offset of a boundary circumferences C_{ij} will become torii $T_{C_{ij}}$ with outer radius the same as the boundary circumference r'_{ij} and the the inner radius the offset r. It may obtained by rotating on the x' axis the circle $(x' - h'_{ij})^2 + (y' - r'_{ij})^2 - r^2 = 0$ of radius r centered at (h'_{ij}, r'_{ij}):

$$(x - h'_{ij})^2 + (\sqrt{y^2 + z^2} - r')^2 - r^2 = 0$$
$$((x - h'_{ij})^2 + y^2 + z^2 - r^2)^2 + 4h'^2_{ij}(z^2 + y^2)$$

3. The r offset of a boundary point will be a sphere centered at the point p_{ijk} and radius r: $S_{p_{ijk}}$

When calculating the outer offset of the union of spheres M_r^+ the offset of the spherical surfaces will dominate in respect to the torii around the intersection circumference and the sphere centered on the intersection points (See Figure **??**).

At this point one has to compute all the components of the outer offset M_r^+. The boundary circles of the surface patch of M_r^+ not only have changed but new intersections may be created. Operating similarly to the original patch, the intersection plane will be:

$$x' = h''_{ij} = \frac{d_{ij}^2 + ((r_j + r)^2 - (r_i + r)^2)}{2d_{ij}}$$

The new intersection circumference $(C_r^+)_{ij}$ will have radius:

$$r''_{ij} = (r_i + r)^2 - h''^2_{ij} = (r_i + r)^2 - \left(\frac{d_{ij}^2 + (r_j + r)^2 - (r_i + r)^2}{2d_{ij}} \right)^2$$

Similarly the intersection points must be recalculated as $(p_r^+)_{ijk}$.

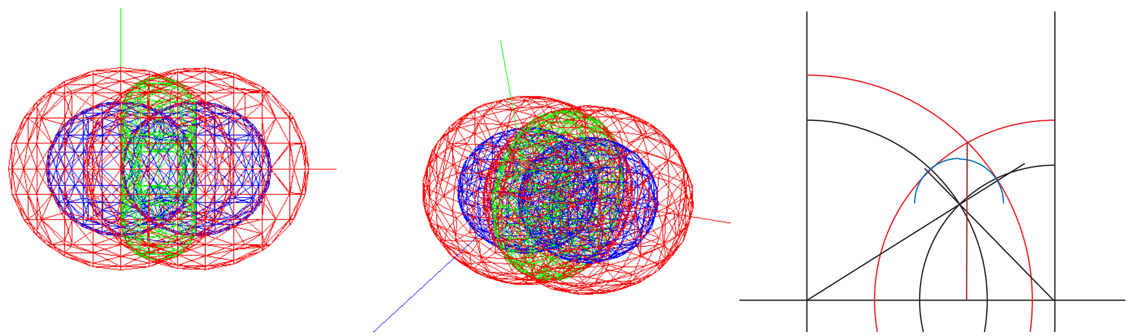


Figure 2: Outer offset components of two spheres

When calculating the inner offset $(M_r^+)_r^-$, the inner surface of the torii $T_{(C_r^+)_{ij}}$ created by the boundary circles (r''_{ij}, h''_{ij}) of the outer offset surface M_r^+ will not be hidden but have the role of "smoothing" out the spheres (See Figure **??**).

The boundary between a spherical surfaces $((S_r^+)_r^-)_i$ and the joining torus $T_{(C_r^+)_{ij}}$ will be a circumference (See Figure **??**). One may calculate the containing plane and its radius

$$\hat{r_{ij}} = \frac{r_i r''_{ij}}{r_i + r}$$

$$x = \hat{h_{ij}} = \frac{r_i h''_{ij}}{r_i + r}$$

Similarly the spheres at the intersection points $(p_r^+)_{ijk}$ will join smoothly the torii (See Figure **??**).

Question 3. Consider a parallel n-stack of n-circles of different radii with one circle per plane, and at possibly a different location (center) in each plane. Provide an A- patch representation of a smooth surface spline which C^1-interpolates the stack of n-circles.

Sol. Firstly, we compute a surface to blend two adjacent circles with some linear constraints.

Let

$$f_1 = (x - x_1)^2 + (y - y_1)^2 - r_1^2; \quad g_1 = z - z_1;$$

$$f_2 = (x - x_2)^2 + (y - y_2)^2 - r_2^2; \quad g_2 = z - z_2;$$

It is easy to prove there is no quadratic surface C^1 interpolate the two circles.

Next we compute if there exists a cubic surface to blend the two circles with some linear constraints.

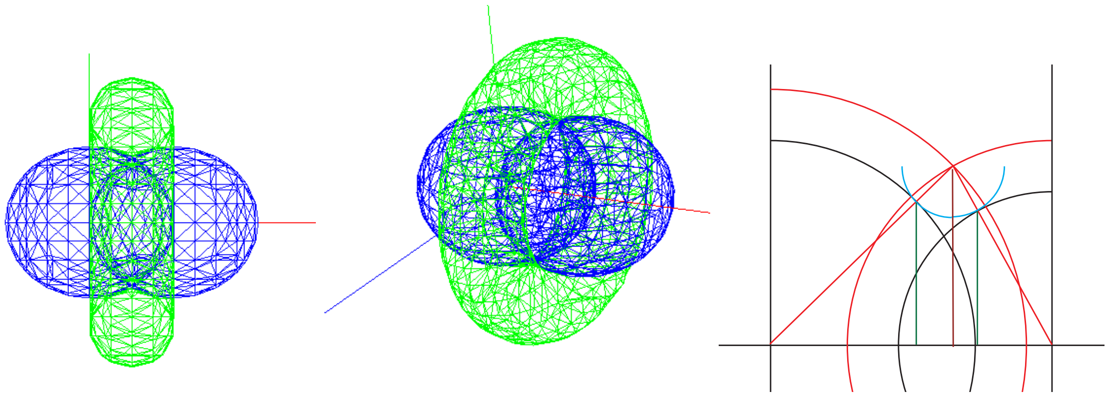


Figure 3: Inner offset components of two spheres

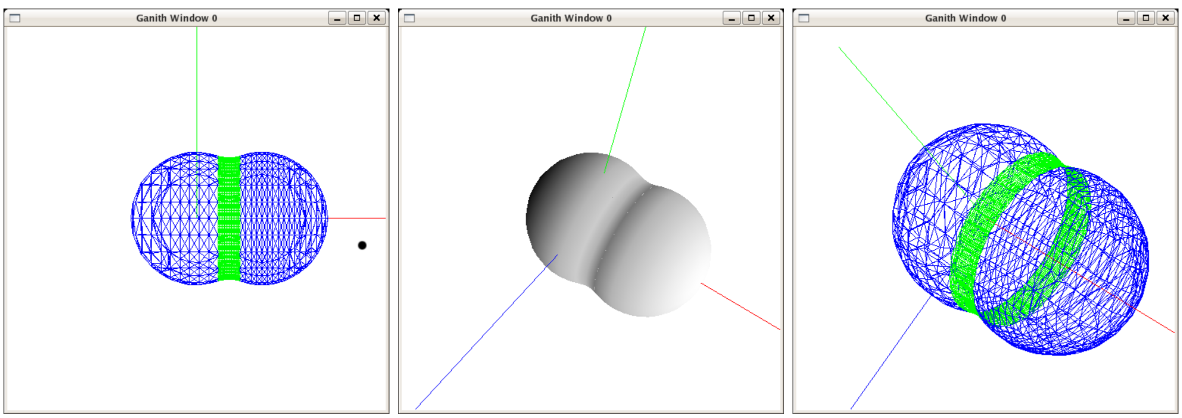


Figure 4: Inner offset of two spheres, clipped at the intersection planes

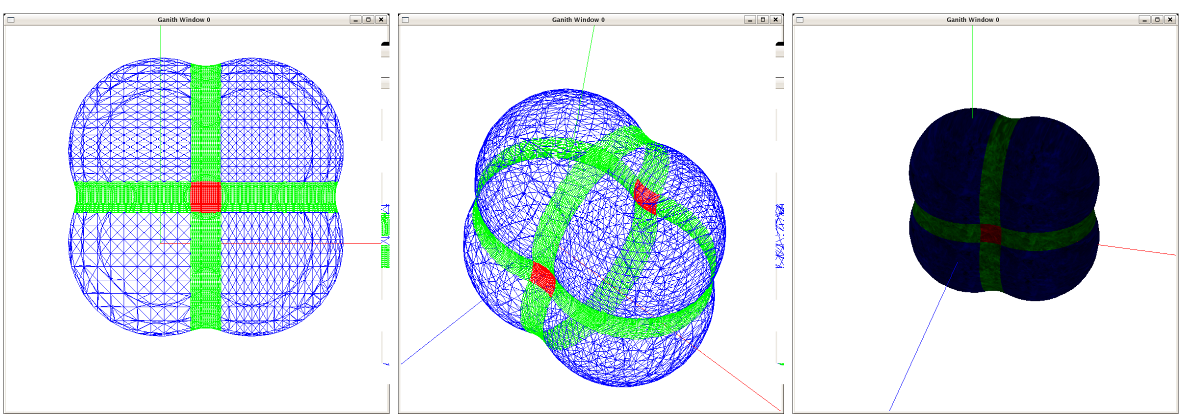


Figure 5: Inner offset four spheres

Let

$$x_{min} = \min\{x_1 - r_1, x_2 - r_2\},\ x_{max} = \max\{x_1 + r_1, x_2 + r_2\},$$

$$y_{min} = \min\{y_1 - r_1, y_2 - r_2\},\ y_{max} = \max\{y_1 + r_1, y_2 + r_2\},$$

$$z_{min} = \min\{z_1, z_2\},\ z_{max} = \max\{z_1, z_2\}.$$

Then the six planes

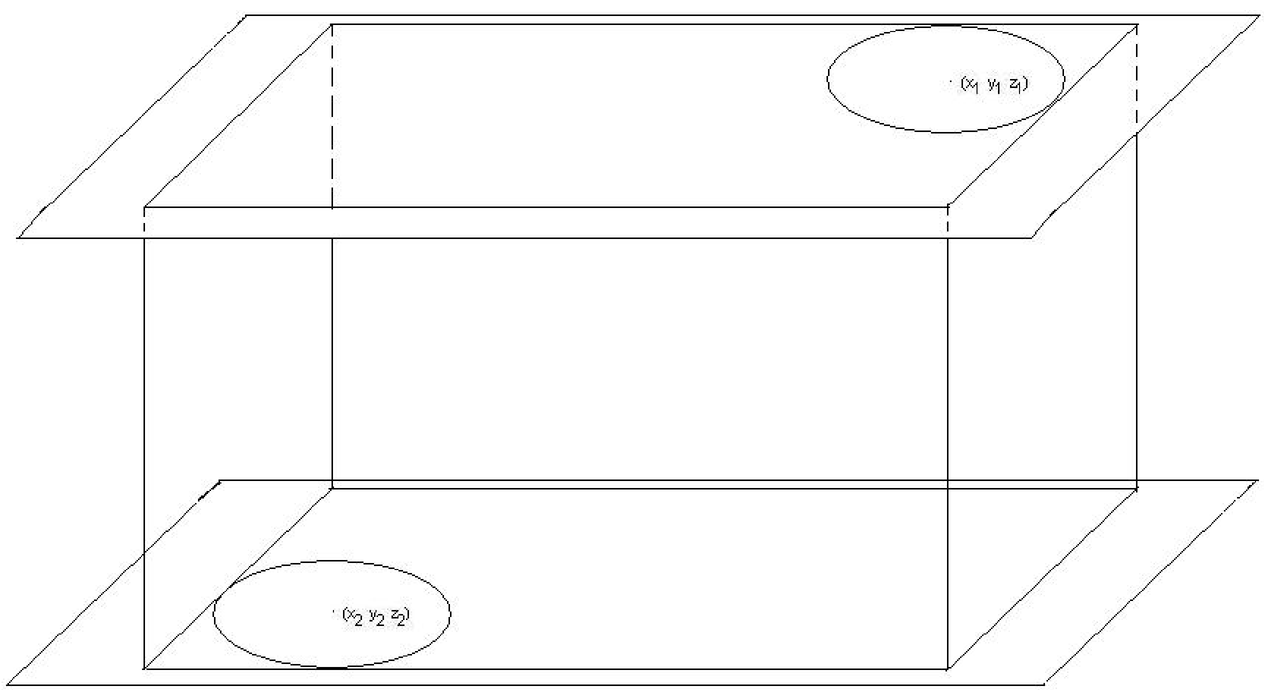
$$HX_1(x, y, z) = x - x_{min}, \quad HX_2(x, y, z) = x - x_{max}, \quad HY_1(x, y, z) = y - y_{min},$$

$$HZ_1(x, y, z) = z - z_{min}, \quad HZ_2(x, y, z) = z - z_{max}, \quad HY_2(x, y, z) = y - y_{max}$$

compose a hexahedron. Each of the planes HX_1, HX_2, HY_1, HY_2 is a tangent plane of one of the circles. We require that the blending surface in this hexahedron. That is to say to find a blending surface with linear constraints:

$$HX_1(x, y, z) > 0, \quad HX_1(x, y, z) < 0, \quad HY_1(x, y, z) > 0, \quad \forall (x, y, z) \in \text{blending surface}$$

$$HZ_1(x, y, z) > 0, \quad HZ_2(x, y, z) < 0, \quad HY_2(x, y, z) < 0, \quad \forall (x, y, z) \in \text{blending surface}$$



Let

$$a_1 = 2z - (z_1 + z_2); \quad b_1 = -2\frac{x_1 - x_2}{z_1 - z_2}x - 2\frac{y_1 - y_2}{z_1 - z_2}y + \frac{(x_1^2 + y_1^2 - r_1^2) - (x_2^2 + y_2^2 - r_2^2)}{z_1 - z_2};$$

$$a_2 = 2z - (z_1 + z_2); \quad b_2 = -2\frac{x_1 - x_2}{z_1 - z_2}x - 2\frac{y_1 - y_2}{z_1 - z_2}y + \frac{(x_1^2 + y_1^2 - r_1^2) - (x_2^2 + y_2^2 - r_2^2)}{z_1 - z_2};$$

Then

$$
\begin{aligned}
f &= a_1 f_1 + b_1 g_1^2 = a_2 f_2 + b_2 g_2^2 \\
&= \left(-\frac{1}{2}\, z_1 - \frac{1}{2}\, z_2\right) x^2 + x^2 z - \frac{(-x_2 + x_1)\, x z^2}{z_1 - z_2} + \left(-2\, x_1 + 2\, \frac{(-x_2 + x_1)\, z_1}{z_1 - z_2}\right) xz \\
&\quad + \left(-2\, \left(-\frac{1}{2}\, z_1 - \frac{1}{2}\, z_2\right) x_1 - \frac{(-x_2 + x_1)\, {z_1}^2}{z_1 - z_2}\right) x + \left(-\frac{1}{2}\, z_1 - \frac{1}{2}\, z_2\right) y^2 + y^2 z \\
&\quad - \frac{(-y_2 + y_1)\, y z^2}{z_1 - z_2} + \left(-2\, y_1 + 2\, \frac{(-y_2 + y_1)\, z_1}{z_1 - z_2}\right) yz + \left(-2\, \left(-\frac{1}{2}\, z_1 - \frac{1}{2}\, z_2\right) y_1 - \frac{(-y_2 + y_1)\, {z_1}^2}{z_1 - z_2}\right) y \\
&\quad + \frac{1}{2}\, \frac{\left({x_1}^2 + {y_1}^2 - {r_1}^2 + {r_2}^2 - {y_2}^2 - {x_2}^2\right) z^2}{z_1 - z_2} \\
&\quad + \left({x_1}^2 + {y_1}^2 - {r_1}^2 - \frac{\left({x_1}^2 + {y_1}^2 - {r_1}^2 + {r_2}^2 - {y_2}^2 - {x_2}^2\right) z_1}{z_1 - z_2}\right) z \\
&\quad + \left(-\frac{1}{2}\, z_1 - \frac{1}{2}\, z_2\right) \left({x_1}^2 + {y_1}^2 - {r_1}^2\right) + \frac{1}{2}\, \frac{\left({x_1}^2 + {y_1}^2 - {r_1}^2 + {r_2}^2 - {y_2}^2 - {x_2}^2\right) {z_1}^2}{z_1 - z_2}
\end{aligned}
$$

is a cubic surface C^1-interpolates the two circles.

We should check if this cubic surface is in the above hexahedron.

If the cubic surface is not in the hexahedron when $z \in (z_{min}, z_{max})$ then it must intersects at least one of the four planes. So we only need to check if any of the equation systems

$$
\begin{cases} f(x, y, z) = 0 \\ HX_1(x, y, z) = 0 \end{cases}
$$

$$
\begin{cases} f(x, y, z) = 0 \\ HX_2(x, y, z) = 0 \end{cases}
$$

$$
\begin{cases} f(x, y, z) = 0 \\ HY_1(x, y, z) = 0 \end{cases}
$$

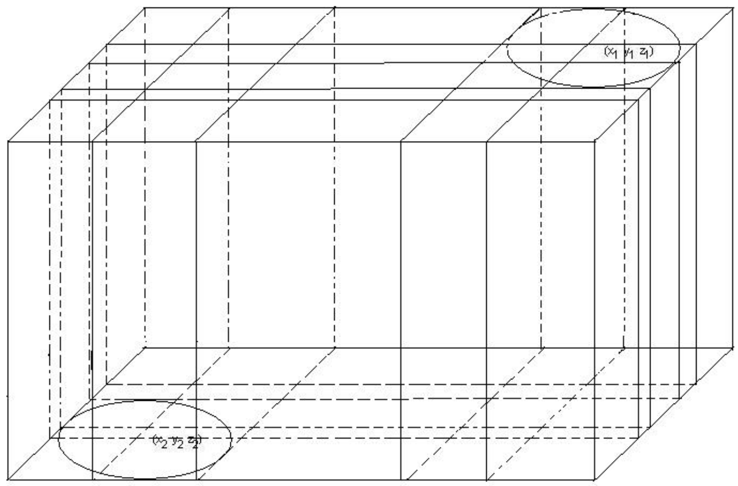
$$
\begin{cases} f(x, y, z) = 0 \\ HY_2(x, y, z) = 0 \end{cases}
$$

has solution in (z_{min}, z_{max}).

If neither of the equation systems has solution then this cubic surface is satisfying all the linear constraint.

Else, we must make the degree of the blending surface higher. We seek for a quartic surface to blend the two circles. Then we can get a family of quartic surfaces. We choose one from them which satisfies all the linear constraints.

When we get the blending surface S, then construct A-patch over this surface.



We define A-patches in tensor domain. Consider the hexahedron H defined by the six planes $V(HX_1), V(HX_2)$, $V(HY_1), V(HY_2), V(HZ_1), V(HZ_2)$. For each circle we partition it to four parts and project the partitions to the other plane (see the figure above). Besides this we continue partitioning H with a set R of hexahedrons s.t. any line segment joining vertices of any chosen hexahedron in R does not intersect S more than once, and, $\mid R \mid$ is the smallest among all possible such sets.

After we get the scaffold we can construct A-patch within each small hexahedron r.

Question 4. Consider a pair of non-parallel planes P_1 and P_2, where the dihedral angle between P_1 and P_2 is less than forty-five degrees. Let there be a single circle C_1 on P_1 and two circles C_2 and C_3 on P_2, all of different radii and at different locations (center) in each plane. Describe a method of generating a family of smooth low degree surface splines which C^1-interpolates (joins) the circles C_1, C_2 and C_3. What is the algebraic and geometric degrees of your surface spline ? What parameter family did you generate ? Provide additionally a construction that describes, either an A-patch or a tensor-product B-spline patch representation of this smooth surface family.

Sol. Let the center of circle C_2 be the origin O, the line passes by the centers of circle C_2 and circle C_3 to be the x-axis, the plan P_2 be X-Y plan. Denote θ the dihedral angle between P_1 and P_2. Then the equations of the two planes are

$$P_1 = x\sin\theta - z\cos\theta$$

$$P_2 = z$$

respectively. The three circles should be defined as

$$C_1 : g_1 = (x\cos\theta + z\sin\theta - x_1)^2 + (y - y_1)^2 - r_1^2,\ P_1 = x\sin\theta - z\cos\theta$$

$$C_2 : g_2 = (x - x_2)^2 + (y - y_2)^2 - r_2^2,\ P_2 = z$$

$$C_3 : g_3 = (x - x_3)^2 + (y - y3)^2 - r_3^2,\ P_3 = z$$

It is easy to prove that there is no cubic surface C^1-interpolating the three circles.

$$
\begin{aligned}
f = \quad & (-b_{2020} + a_{3002} + b_{3020} + \frac{1}{\cos(\theta)}(-\sin(\theta)\, b_{1101} + \sin(\theta)\, b_{1101}\, (\cos(\theta))^2 + (\cos(\theta))^3\, b_{3020} \\
& + (\cos(\theta))^3\, a_{3002} + b_{2020}\, \cos(\theta) - \cos(\theta)\, a_{3002} - b_{3020}\, \cos(\theta) + \cos(\theta)\, b_{1002} - (\cos(\theta))^3\, b_{1002}))z^2x^2 \\
& +2\, \sin(\theta)\, (\cos(\theta)\, a_{3002} - \cos(\theta)\, b_{1002} + b_{3020}\, \cos(\theta) + \sin(\theta)\, b_{1101})\, xz^3 - 2\, \cos(\theta)\, x_1\, b_{3020} xz^2 \\
& + (-2\, (-b_{2020} + a_{3002} + b_{3020})\, x_2 + 2\, a_{3002}\, x_2 - 2\, x_2\, b_{2020} + 2\, x_2\, b_{3020} - 2\, \cos(\theta)\, x_1\, a_{3002})\, xz^2 \\
& + (-2\, (-b_{2020} + a_{3002} + b_{3020})\, y_2 + 2\, y_2\, a_{3002} - 2\, y_2\, b_{2020} + 2\, y_2\, b_{3020} - 2\, a_{3002}\, y_1 - 2\, y_1\, b_{3020})\, z^2 y \\
& + \left(-b_{1101}\, \cos(\theta)\sin(\theta) + b_{1002}\, (\cos(\theta))^2 - b_{3020}\, (\cos(\theta))^2 - a_{3002}\, (\cos(\theta))^2 + b_{3020} + a_{3002}\right) z^4 \\
& + (b_{3020} + a_{3002})\, z^2y^2 - 2\, \sin(\theta)\, x_1\, (b_{3020} + a_{3002})\, z^3 \\
& +((-b_{2020} + a_{3002} + b_{3020})\left({x_2}^2 + {y_2}^2 - {r_2}^2\right) - a_{3002}\, {x_2}^2 + a_{3002}\, {r_2}^2 - a_{3002}\, {y_2}^2 + b_{2020}\, {x_2}^2 \\
& + b_{2020}\, {y_2}^2 - b_{2020}\, {r_2}^2 - b_{3020}\, {x_2}^2 - b_{3020}\, {y_2}^2 + b_{3020}\, {r_2}^2 + {x_1}^2 a_{3002} + {y_1}^2 a_{3002} - {r_1}^2 a_{3002} \\
& + {y_1}^2 b_{3020} - {r_1}^2 b_{3020} + {x_1}^2 b_{3020})z^2
\end{aligned}
$$

is a family of quartic surfaces which C^1-interpolate the three cubic surfaces, where a, b are free parameters.

Next we give another method to calculate the quartic surface which C^1-interpolates (joins) the circles C_1, C_2 and C_3[**?**].

Since the circle is of degree 2, with Bezout theorem, if the quartic surface and the circle have 8 common points, then the quartic surface must contain the whole circle. Select 8 points $P_{i,j} \quad (j = 1, \cdots, 8)$ from each of the circle $C_i \quad (i = 1, 2, 3)$ and substitute them into the expression of the quartic surface

$$f(P_{i,j}) = 0, j = 1, \cdots, 8, \quad i = 1, 2, 3.$$

Then we get 24 homogeneous linear equations.

The normal of the circle be $n(x, y, z) = (n_x(x, y, z), n_y(x, y, z), n_z(x, y, z))$ where n_x, n_y, and n_z are polynomials of maximum degree 1. So for tangent condition we need another 8 conditions for each circle.

(a) Compute $t(t_x, t_y, t_z) = \nabla g_i(x, y, z) x \nabla P_i(x, y, z)$. Note $t = (t_x, t_y, t_z)$is the tangent vector to C.

(b) Select one of the following:

i If $t_x \neq 0$, use the equation $f_y \cdot n_z - n_y \cdot f_z = 0$.

ii If $t_y \neq 0$, use the equation $f_x \cdot n_z - n_x \cdot f_z = 0$.

iii If $t_z \neq 0$, use the equation $f_x \cdot n_y - n_x \cdot f_y = 0$.

Substitute each point $P_{ij} \quad (j = 1, \cdots, 8,\ i = 1, 2, 3)$ into the above-selected equation to yield 24 additional homogeneous linear equations in the coefficients of f(x, y, z). Now we have 48 homogeneous linear equations

$$MX = 0$$

where $M \in \mathbb{R}^{48\times35}$ is a matrix of the linear equations, and $X \in \mathbb{R}^{35}$ is the vector whose elements are unknown coefficients of surface $f(x, y, z)$.

In order to solve the linear system in a computationally stable manner, we compute the singular value decomposition (SVD) of M. Hence, M is decomposed as $M = U\Sigma V^T$ where $U \in \mathbb{R}^{48\times48}$ and $V \in \mathbb{R}^{35\times35}$ are orthonormal matrices, and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_s) \in \mathbb{R}^{48\times35}$ is a diagonal matrix with diagonal elements $\sigma_l \geq \sigma_2 \geq \cdots \geq \sigma_s \geq 0$. It can be proved that the rank r of M is the number of the positive diagonal elements of Σ, and that the last $35Cr$ columns of V span the nullspace of M. (If $r \geq 35$ then it means there exists no quartic surface to Hermite interpolate the three circles. For this cases we know it exits.) Hence, the nontrivial solutions of the homogeneous linear system are compactly expressed as $\left\{x(\neq 0) \in \mathbb{R}^{35} \mid x = \Sigma_{i=1}^{35-r}\omega_i \cdot V_{r+i)}, \mathrm{where}\omega_\mathrm{i} \in \mathbb{R}, \mathrm{and} \mathrm{V_j} \mathrm{is the j-th column of V}\right\}$, or $x = V_{35-r}W$, where $V_{35-r} \in R^{35\times(35-r)}$ is made of the last $35-r$ columns of V, and W is a $35-r$ vector for free parameters.

From the family of quartic surfaces we need to choose one which is in a given box.

Define

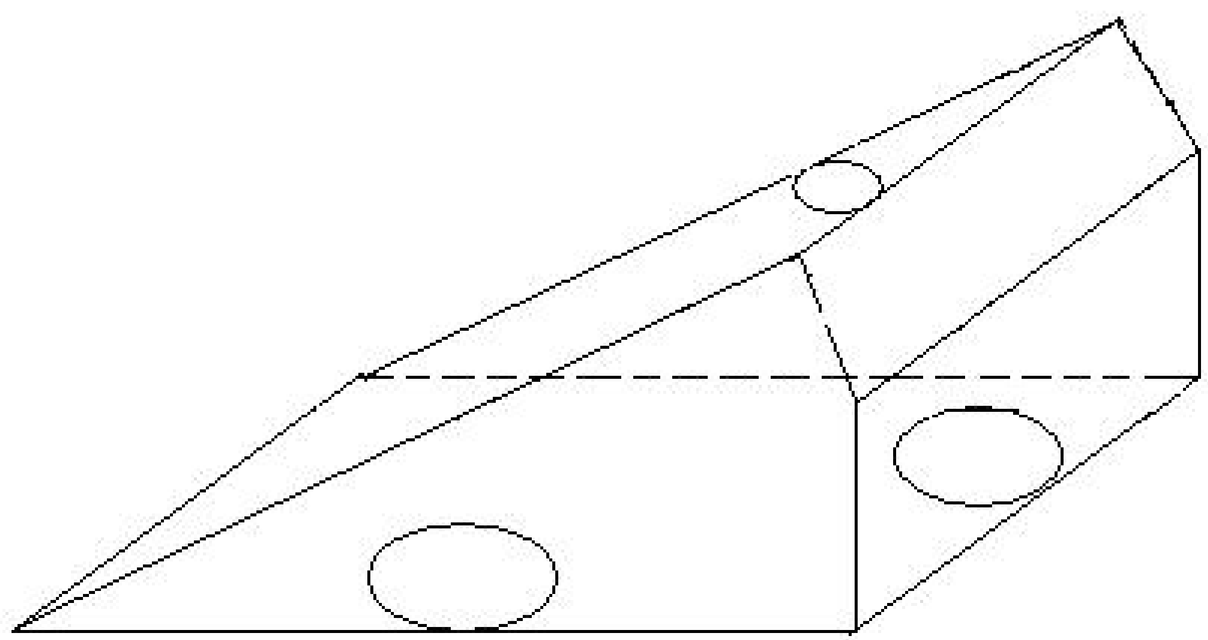
$$x_{max} = \max\{x_2 + r_2, x_3 + r_3\},$$

$$y_{min} = \min\{y_1 - r_1, y_2 - r_2, y_3 - r_3\}, \quad y_{max} = \max\{y_1 + r_1, y_2 + r_2, y_3 + r_3\}.$$

Define four planes

$$HX = x - x_{max}, \quad HY_1 = y - y_{min}, \quad HY_2 = y - y_{max}, \quad HXZ = x\cos\theta + z\sin\theta - (x_1 + r_1).$$

Each of the planes HX, HXZ, HY_1, HY_2 is a tangent plane of one of the circles. These four planes with the planes P_1 and P_2 together construct a box.



Then we choose one quartic surface from the family of surfaces we obtained by SVD method which have no intersection with any of the planes $V(HX)$, $V(HXZ)$, $V(HY_1)$, $V(HY_2)$, $V(P_1)$, $V(P_2)$ during the corresponding intervals.

Next one construct A-patch representation.

Consider the hexahedron H defined by the six planes $V(HX), V(HXZ), V(HY_1), V(HY_2), V(HZ_1), V(HZ_2)$. For each circle we partition it to four parts and project the partitions to the other plane. Besides this we continue partitioning H with a set R of hexahedrons s.t. any line segment joining vertices of any chosen hexahedron in R does not intersect S more than once, and, $\mid R \mid$ is the smallest among all possible such sets.

After we get the scaffold we can construct A-patch within each small hexahedron r.

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