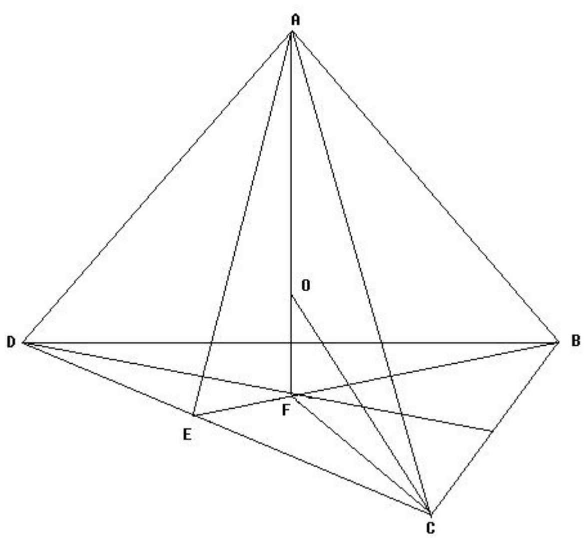
Solutions to Exercise 5: Geometric Modeling

CS 384R, CAM 395T, BME 385J: Fall 2007

Nov. 15, 2007

Question 1. One measure of a quality tetrahedron T is the aspect ratio bound, γ, where γ = ratio of the radii of the circumscribing sphere of T and the inscribed sphere of T . What is γ for a regular tetrahedron?

Sol. Let the length of each edge of the regular tetrahedron is a. It is easy to see that the centers of inscribing sphere and the one of the circumscribing sphere is the same point O, and it is on the line AF, where AF is perpendicular to the plane ABC.

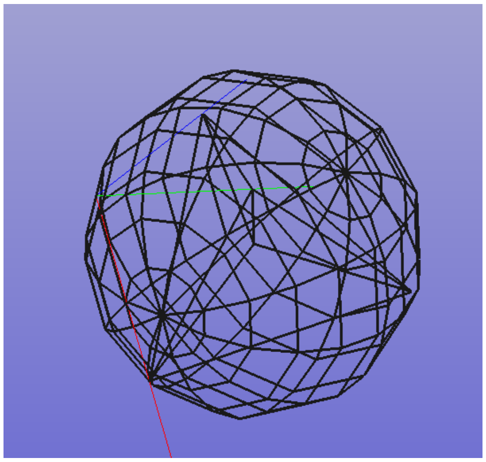
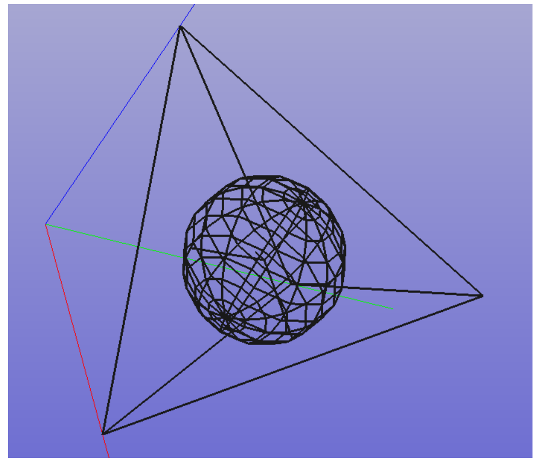


$AE = \frac{\sqrt{3}}{2}a, \quad EF = \frac{\sqrt{3}}{6}a, \quad AF = \sqrt{AE^2 - EF^2} = \frac{\sqrt{6}}{3}a, \quad CE = \frac{a}{2}, \quad CF = \sqrt{EF^2 + CE^2} = \frac{\sqrt{3}}{3}a.$

$OA = OC, \quad OC + OF = OA + OF = AF = \frac{\sqrt{6}}{3}a$ and $OC^2 = OF^2 + CF^2$, so we get

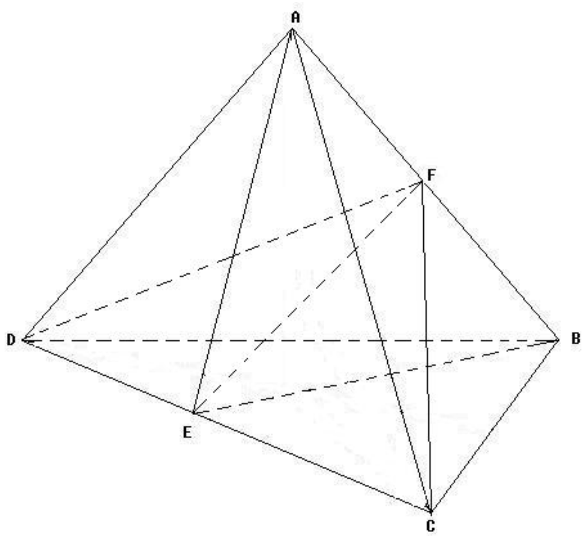
$$OF = \frac{\sqrt{6}}{12}a, \quad OC = \frac{\sqrt{6}}{4}a.$$

$$\gamma = OC/OF = 3.$$

Question 2. For a tetrahedron T, consider the mid-edge decomposition of T, which splits T into four-sub tetrahedra. If T is initially a regular tetrahedron, what is γ for the four-sub tetrahedra of T under mid-edge subdivision?

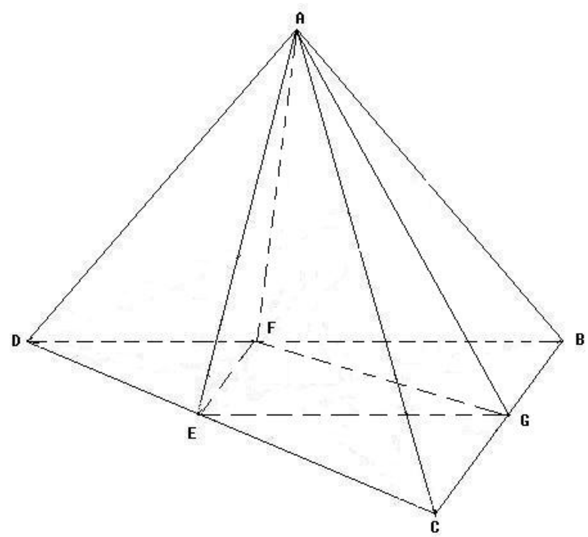
Sol. Connect the midpoint E of edge CD and the midpoint F of edge AB, and add edges AE, BE, CF and DF, then we get four sub-tetrahedra: $AEFC$, $BEFC$, $AEFD$ and $BEFD$. All the tetrahedra are the same up to an affine transformation, therefore we consider only one of them.



For a regular tetrahedron with the length of each edge be a, we can set the four vertices to be A: $(\frac{1}{\sqrt{2}}a, 0, 0)$,B:$(0, \frac{1}{\sqrt{2}}a, 0)$,C:$(0, 0, \frac{1}{\sqrt{2}}a)$, D:$(\frac{1}{\sqrt{2}}a, \frac{1}{\sqrt{2}}a, \frac{1}{\sqrt{2}}a)$. Then the mid-points be E:$(\frac{1}{2\sqrt{2}}a, \frac{1}{2\sqrt{2}}a, \frac{1}{\sqrt{2}}a)$, F:$(\frac{1}{2\sqrt{2}}a, \frac{1}{2\sqrt{2}}a, 0)$.

For the tetrahedron ACEF, the center of the circumscribing sphere is $(\frac{\sqrt{2}}{8}a, -\frac{\sqrt{2}}{8}a, \frac{\sqrt{2}}{8}a)$, and radius is $\frac{\sqrt{22}}{8}a$. The center of the circumscribing sphere is $(\frac{\sqrt{2}}{4}a, \frac{3\sqrt{2}}{4} - \frac{\sqrt{3}}{2}a, \frac{\sqrt{2}}{4}a)$, and radius is $-\frac{1}{2} + \frac{\sqrt{6}}{4}a$.

$$\gamma = \frac{\frac{\sqrt{22}}{8}a}{-\frac{1}{2}+\frac{\sqrt{6}}{4}a} \approx 5.21748919.$$



For a regular tetrahedron with the length of each edge be a, we can set the four vertices to be A: $(\frac{1}{\sqrt{2}}a, 0, 0)$,B:$(0, \frac{1}{\sqrt{2}}a, 0)$,C:$(0, 0, \frac{1}{\sqrt{2}}a)$, D:$(\frac{1}{\sqrt{2}}a, \frac{1}{\sqrt{2}}a, \frac{1}{\sqrt{2}}a)$. Then the mid-points be E:$(\frac{1}{2\sqrt{2}}a, \frac{1}{2\sqrt{2}}a, \frac{1}{\sqrt{2}}a)$, F:$(\frac{1}{2\sqrt{2}}a, \frac{1}{\sqrt{2}}a, \frac{1}{2\sqrt{2}}a)$, G:$(0, \frac{1}{2\sqrt{2}}a, \frac{1}{2\sqrt{2}}a)$.

The sub-tetrahedron ADEF, ACEG, ABFG are similar to each other.We only consider ABFG. The center of the circumscribing sphere is $(\frac{\sqrt{2}}{4}a, \frac{\sqrt{2}}{4}a, 0)$, and the radius is $\frac{1}{2}a$. The center of the inscribed sphere is $(\frac{15\sqrt{(2)}-\sqrt{(66)}}{64}a, \frac{17\sqrt{2}+\sqrt{66}}{64}a, \frac{15\sqrt{2}-\sqrt{66}}{64}a)$, the radius is $\frac{15\sqrt{6}-3\sqrt{22}}{192}a$. So

$$\gamma = \frac{1/2a}{\frac{15\sqrt{6}-3\sqrt{22}}{192}a} \approx 4.23446612$$

For the tetrahedron AEFG, the center of the circumscribing sphere is $(\frac{5\sqrt{2}}{16}a, \frac{3\sqrt{2}}{16}a, \frac{3\sqrt{2}}{16}a)$, and the radius is $\frac{2\sqrt{6}}{16}a$. The center of the circumscribing sphere is $(\frac{15\sqrt{2}+\sqrt{66}}{96}a, \frac{33\sqrt{2}-\sqrt{66}}{96}a, \frac{33\sqrt{2}-\sqrt{66}}{96}a)$, and radius is $\frac{3\sqrt{22}-\sqrt{6}}{96}a$.

$$\gamma = \frac{\frac{2\sqrt{6}}{16}a}{\frac{3\sqrt{22}-\sqrt{6}}{96}a} \approx 3.793816488.$$

Question 3. For a tetrahedron T, consider choosing a point p inside of T, which if joined to the four vertices of T yields a 4-way split of T into sub-tetrahedra. Which of the following choices of p yields the best γ split of T : (a) p is the center of the circumscribing sphere of T? (b) p is the center of the inscribed sphere of T ? (iii) p is the centroid of T?

Sol. Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ be the four vertices, then the coordinates of the center of the circumscribing sphere are given by

$$x = \frac{N_1}{2D_3}, \quad y = \frac{N_2}{2D_3}, \quad Z = \frac{N_3}{2D_3},$$

where

$$N_1 = \begin{vmatrix} x_2^2 + y_2^2 + z_2^2 - (x_1^2 + y_1^2 + z_1^2) & (y_2 - y_1) & (z_2 - z_1) \\ x_3^2 + y_3^2 + z_3^2 - (x_1^2 + y_1^2 + z_1^2) & (y_3 - y_1) & (z_3 - z_1) \\ x_4^2 + y_4^2 + z_4^2 - (x_1^2 + y_1^2 + z_1^2) & (y_4 - y_1) & (z_4 - z_1) \end{vmatrix}$$

$$N_2 = \begin{vmatrix} (x_2 - x_1) & x_2^2 + y_2^2 + z_2^2 - (x_1^2 + y_1^2 + z_1^2) & (z_2 - z_1) \\ (x_3 - x_1) & x_3^2 + y_3^2 + z_3^2 - (x_1^2 + y_1^2 + z_1^2) & (z_3 - z_1) \\ (x_4 - x_1) & x_4^2 + y_4^2 + z_4^2 - (x_1^2 + y_1^2 + z_1^2) & (z_4 - z_1) \end{vmatrix}$$

$$N_3 = \begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) & x_2^2 + y_2^2 + z_2^2 - (x_1^2 + y_1^2 + z_1^2) \\ (x_3 - x_1) & (y_3 - y_1) & x_3^2 + y_3^2 + z_3^2 - (x_1^2 + y_1^2 + z_1^2) \\ (x_4 - x_1) & (y_4 - y_1) & x_4^2 + y_4^2 + z_4^2 - (x_1^2 + y_1^2 + z_1^2) \end{vmatrix}$$

and

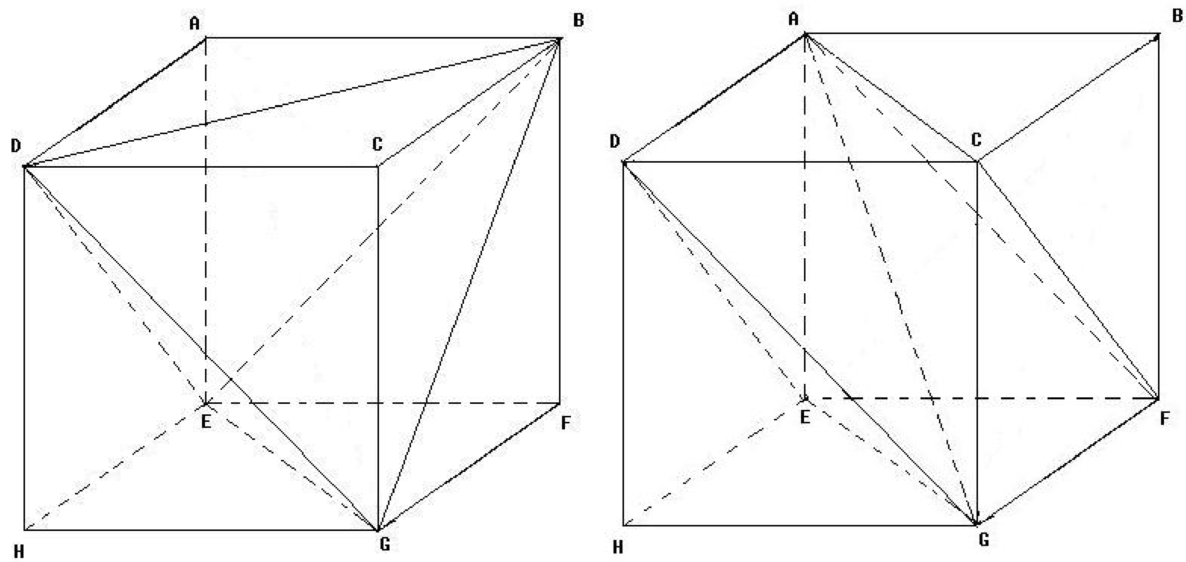
$$D_3 = \begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \\ (x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1) \end{vmatrix}$$

If the center of the circumscribing sphere is located in the tetrahedron, then we choose p to be this center, which yields the best γ split of T.

If the center of the circumscribing sphere is not in the tetrahedron, then we project this point to the nearest face of the tetrahedron, and set the projection point to be p, then it yields the best γ split of T.

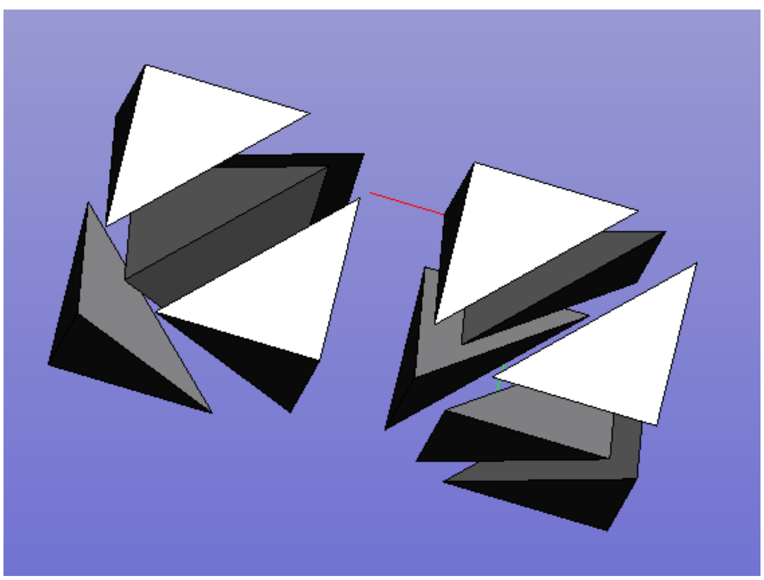
Question 4. Describe two ways to decompose a cube into tetrahedra, without using any Steiner points (i.e. no additional vertices other than the original vertices). Which decomposition yields better γ for the resulting tetrahedra ?

Sol. For the first method, we get tetrahedra: $ABDE$, $BCDG$, $DEGH$,$BEGF$ and $BDEG$. Notice that the tetrahedron $BCDG$ is regular with all edges equal $\sqrt{2}$ and the other are rectangular with three edges equal to 1 and one $\sqrt{2}$.



For the second method, we get tetrahedra: $ABCF$,$ACFG$,$AEFG$ $ACDG$,$ADEG$,$DEGH$.

Notice that in the first case the The first method yields better γ for the resulting tetrahedra.



Question 5. How many ways are there to decompose an octahedron, an icosahedron, and a dodechadron into tetrahedra without Steiner points. ?

Sol.

Question 6. One measure of a quality quad element Q or hex element H (also called a hexahedron or brick element) is the Jacobian norm J. Assume $x \in \mathcal{R}^2$ is a vertex of the quad or a hex, and $x_i \in \mathcal{R}^3$ for $i = 1, \cdots, m$

are its neighboring vertices, where $m = 2$ for a quad and $m = 3$ for a hex. Edge vectors are defined as $e_i = x_i - x$ with $i = 1, \cdots, m$, and the Jacobian norm is $\det([e_1, \cdots, e_m])$. (a) What is J for the unit square and the unit cube ? (b)When is J zero for a quad or a hex ? (c) When is J negative for a quad or hex ?

Sol. (a) The $[e_1, \cdots, e_m]$ for the unit square is the identity matrix. The Jacobian norm $J = 1$:

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

(b) If $J = 0$ then $[e_1, \cdots, e_m]$ is not of full rank and a vector e_i may be expresses as a linear combination of the other $(m-1)$: $e_i = \sum_{j \neq i} a_j e_j$.

(c) In the bi-dimentional case the signed area is positive if the vectors e_1 and e_2 are ordered counterclockwise and the the angle is less then π. If the original quad had its vertexes ordered then a negative J means an angle at the vertex greater then π.

In the tri-dimentional one may define three kind of angles:

face Three angles on each face between edges e_i and e_j.

edge Three dihedral angles on each edge e_i between two faces.

vertex Solid angle on the vertex x between all three edges e_1, e_2, e_3.

If the vertexes x_i are consistently ordered the sign of the Jacobian norm is negative if the face and edge angles are less then pi and the vertex angle is less then 2π.

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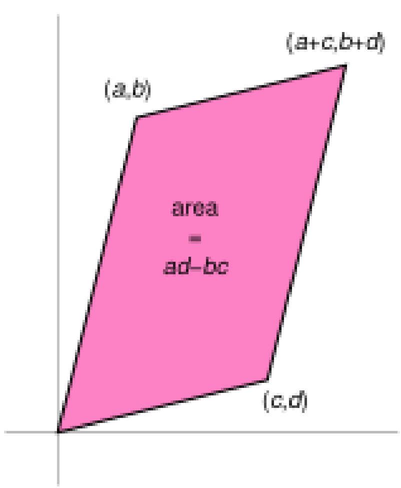

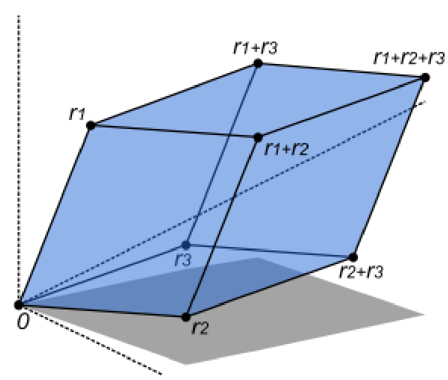

Question 7. Given a collection of n disjoint triangles T_i of different sizes within a bounding rectangle D, describe a method to decompose the region bounded by (D - union of all T_i) into quadrilaterals of nice quality i.e. decompose the region inside D but outside each triangle T_i into quads, all with positive Jacobian norms J.

Sol. Build the Voronoi diagram using triangle vertexes ...

One triangle in a quad. Build Vonoroi diagram. Connect each vertex to nearest side of enclosing quad.