Solutions for Sample Final Exam - CS 313k

1. (a) B(x): x is a baby, I(x): x is illogical
   $\forall x[B(x) \rightarrow I(x)]$
   (b) J(x): x dislikes Joe, S(x): Susan likes x
   $\forall x[J(x) \rightarrow S(x)]$

2. **Proof:** Let x and y be arbitrary real numbers. Let $z = \frac{y-x}{2}$. Since x, y are real, z is also real. So
   $x + z = x + \frac{y-x}{2} = \frac{2x+y-x}{2} = \frac{y+x}{2} = \frac{2y-(y-x)}{2} = y - \frac{y-x}{2} = y - z.$

3. The Venn diagram will be shown in the review session or office hours.

4. Direct proof
   Givens: $n > 2$
   Goal: $n^2 - 1$ is not prime
   Proof: Let n be an arbitrary positive integer. Assume $n > 2$. Then $n^2 - 1 = (n-1)(n+1)$, and since $n > 2$, $n - 1 > 1$ and $n + 1 > 3 > 1$. So $n^2 - 1$ can be written as the product of two integers, neither of which is 1. Therefore $n^2 - 1$ is composite, or not prime.

5. **Proof:** (by induction)
   Base case (n = 1): $2^{3(1)} - 1 = 8 - 1 = 7 = 1(7)$, so $7|2^{3(1)} - 1$.

   Induction hypothesis: Let $k \geq 1$ be arbitrarily chosen. Assume that $7|(2^{3k} - 1)$, i.e., $2^{3k} - 1 = 7r$ for some integer r.
   We must show that $7|(2^{3k+3} - 1)$.

   $2^{3k+3} - 1 = 8(2^{3k}) - 1$
   $= 8(7r + 1) - 1$ for some integer r by the induction hypothesis
   $= 7(8r) + 8 - 1$
   $= 7(8r) + 7$
   $= 7(8r + 1)$, where $8r + 1$ is an integer since r is an integer.
   So $7|(2^{3k+3} - 1)$.

6. Modify the definition of the set A so that $A = \{(i, j)|i, j \in \mathbb{Z}^+\}$.

   (a) First we show R is reflexive. Let $(a, b) \in A$ be arbitrary. Since $ab = ab$, $(a, b)R(a, b)$. So R is reflexive.

   Next we show that R is symmetric. Let $(a, b), (c, d) \in A$ be arbitrarily chosen. Assume $(a, b)R(c, d)$. So $ad = cb$. So $cb = ad$, and so $(c, d)R(a, b)$. Thus R is symmetric.

Next we show that R is transitive. Let $(a,b),(c,d),(e,f) \in A$ be arbitrarily chosen. Assume that $(a,b)R(c,d)$ and $(c,d)R(e,f)$. So $ad = cb$ and $cf = ed$. So $a = \frac{cb}{d}$ and $f = \frac{ed}{c}$. Then $af = \frac{cb}{d}(\frac{ed}{c}) = be = eb$, and so $(a,b)R(e,f)$. So R is transitive.

So R is an equivalence relation.

(b) $[(1,1)] = \{(n,n)|n \in \mathbb{Z}^+\}$

7. (a) First we show that S is reflexive. Let $x \in \mathbb{R}$ be arbitrary. Then $x - x = 0$, and 0 is a non-negative integer, and $0 = 2(0)$ is even. So S is reflexive.

   Now we show that S is anti-symmetric. Let $x, y \in \mathbb{R}$ be arbitrary. Assume xSy and ySx. So $x - y$ is a non-negative even integer and $y - x$ is a non-negative even integer. Since $y - x = -(x - y)$, and $x - y$ and $y - x$ are both non-negative, $x - y = 0 = y - x$. So $y = x$. Therefore S is anti-symmetric.

   Now we show that S is transitive. Let $x, y, z \in \mathbb{R}$ be arbitrary. Assume xSy and ySz. So $x - y \in 2\mathbb{Z}^{\geq 0}$ and $y - z \in 2\mathbb{Z}^{\geq 0}$. Then $x - y = 2k$ for some $k \geq 0$ and $y - z = 2j$ for some $j \geq 0$. Then $x - z = (x - y) + (y - z) = 2k + 2j = 2(k + j)$, where $k + j$ is a non-negative integer since k, j are non-negative integers. So S is transitive.

   (b) No. Note that 5-4 = 1, which is not even, and 4-5 = -1, which is not non-negative. So $(5,4)$ is not in S and $(4,5)$ is not in S. So S is not a total order.

   (c) No minimal elements

8. Redefine $f(x) = 2x + 1$.

   (a) Yes, f is a bijection. First we show that f is 1-1. Let $x, y \in \mathbb{N}$ be arbitrary. Assume $f(x) = f(y)$, i.e., $2x + 1 = 2y + 1$, or $2x = 2y$, or $x = y$. So f is 1-1.

   Next we show that f is onto P. Let $y \in P$ be arbitrary. By definition of P, $y = 2n + 1$ for some natural number n. Then $f(n) = 2n + 1 = y$. So f is onto.

   So f is a bijection.

   (b) $f(\{1,3,4\}) = \{3,7,9\}$

   (c) The preimage of 11 is 5.