

CS341 Review Problems – May 4, 2003
Turing Machines and (Un)Decidability
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1. For each of the following languages, state whether each language is (I) recursive, (II) recursively enumerable but not recursive, or (III) not recursively enumerable. Prove your answer.

- $L_1 = \{\langle M \rangle \mid M \text{ is a TM and there exists an input on which } M \text{ halts in less than } |\langle M \rangle| \text{ steps}\}.$
- $L_2 = \{\langle M \rangle \mid M \text{ is a TM and } |L(M)| \leq 3\}.$
- $L_3 = \{\langle M \rangle \mid M \text{ is a TM and } |L(M)| \geq 3\}.$
- $L_4 = \{\langle M \rangle \mid M \text{ is a TM that accepts all even numbers}\}.$
- $L_5 = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is finite}\}.$
- $L_6 = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is infinite}\}.$
- $L_7 = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is countable}\}.$
- $L_8 = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is uncountable}\}.$
- $L_9 = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are two TMs, and } \varepsilon \in L(M_1) \cup L(M_2)\}.$
- $L_{10} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are two TMs, and } \varepsilon \in L(M_1) \cap L(M_2)\}.$
- $L_{11} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are two TMs, and } \varepsilon \in L(M_1) \setminus L(M_2)\}.$
- $L_{12} = \{\langle M \rangle \mid M \text{ is a TM, } M_0 \text{ is a TM that halts on all inputs, and } M_0 \in L(M)\}.$
- $L_{13} = \{\langle M \rangle \mid M \text{ is a TM, } M_0 \text{ is a TM that halts on all inputs, and } M \in L(M_0)\}.$
- $L_{14} = \{\langle M, x \rangle \mid M \text{ is a TM, } x \text{ is a string, and there exists a TM, } M', \text{ such that } x \notin L(M) \cap L(M')\}.$
- $L_{15} = \{\langle M \rangle \mid M \text{ is a TM, and there exists an input on which } M \text{ halts within 1000 steps}\}.$
- $L_{16} = \{\langle M \rangle \mid M \text{ is a TM, and there exists an input whose length is less than 100, on which } M \text{ halts}\}.$
- $L_{17} = \{\langle M \rangle \mid M \text{ is a TM, and } M \text{ is the only TM that accepts } L(M)\}.$
- $L_{18} = \{\langle k, x, M_1, M_2, \dots, M_k \rangle \mid k \text{ is a natural number, } x \text{ is a string, } M_i \text{ is a TM for all } 1 \leq i \leq k, \text{ and at least } k/2 \text{ TM's of } M_1, \dots, M_k \text{ halt on } x\}.$
- $L_{19} = \{\langle M \rangle \mid M \text{ is a TM, and } |M| < 1000\}.$
- $L_{20} = \{\langle M \rangle \mid \exists x, |x| \equiv_5 1, \text{ and } x \in L(M)\}.$
- $L_{21} = \{\langle M \rangle \mid M \text{ is a TM, and } M \text{ halts on all palindromes}\}.$
- $L_{22} = \{\langle M \rangle \mid M \text{ is a TM, and } L(M) \cap \{a^{2^n} \mid n \geq 0\} \text{ is empty}\}.$
- $L_{23} = \{\langle M, k \rangle \mid M \text{ is a TM, and } |\{w \in L(M) : w \in a^*b^*\}| \geq k\}.$
- $L_{24} = \{\langle M \rangle \mid M \text{ is a TM that halts on all inputs and } L(M) = L' \text{ for some undecidable language } L'\}.$
- $L_{25} = \{\langle M \rangle \mid M \text{ is a TM, and } M \text{ accepts (at least) two strings of different lengths}\}.$
- $L_{26} = \{\langle M \rangle \mid M \text{ is a TM such that both } L(M) \text{ and } \overline{L(M)} \text{ are infinite}\}.$
- $L_{27} = \{\langle M, x, k \rangle \mid M \text{ is a TM, and } M \text{ does not halt on } x \text{ within } k \text{ steps}\}.$
- $L_{28} = \{\langle M \rangle \mid M \text{ is a TM, and } |L(M)| \text{ is prime}\}.$
- $L_{29} = \{\langle M \rangle \mid \text{there exists } x \in \Sigma^* \text{ such that for every } y \in L(M), xy \notin L(M)\}.$
- $L_{30} = \{\langle M \rangle \mid \text{there exist } x, y \in \Sigma^* \text{ such that either } x \in L(M) \text{ or } y \notin L(M)\}.$
- $L_{31} = \{\langle M \rangle \mid \text{there exists a TM } M' \text{ such that } \langle M \rangle \neq \langle M' \rangle \text{ and } L(M) = L(M')\}.$
- $L_{32} = \{\langle M_1, M_2 \rangle \mid L(M_1) \preceq L(M_2)\}.$

- $L_{33} = \{\langle M \rangle \mid M \text{ does not accept any string } w \text{ such that } 001 \text{ is a prefix of } w\}$.
- $L_{34} = \{\langle M, x \rangle \mid M \text{ does not accept any string } w \text{ such that } x \text{ is a prefix of } w\}$.
- $L_{35} = \{\langle M, x \rangle \mid x \text{ is prefix of } \langle M \rangle\}$.
- $L_{36} = \{\langle M_1, M_2, M_3 \rangle \mid L(M_1) = L(M_2) \cup L(M_3)\}$.
- $L_{37} = \{\langle M_1, M_2, M_3 \rangle \mid L(M_1) \subseteq L(M_2) \cup L(M_3)\}$.
- $L_{38} = \{\langle M_1 \rangle \mid \text{there exist two TM's } M_2 \text{ and } M_3 \text{ such that } L(M_1) \subseteq L(M_2) \cup L(M_3)\}$.
- $L_{39} = \{\langle M, w \rangle \mid M \text{ is a TM that accepts } w \text{ using at most } 2^{|w|} \text{ squares of its tape}\}$.

2. If $A \preceq B$ and B is a regular language, does that imply that A is a regular language?
3. Recall the language $A_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM, and } M \text{ accepts } w\}$. Consider the language

$$J = \{w \mid w = 0x \text{ for some } x \in A_{TM} \text{ or } w = 1y \text{ for some } y \in \overline{A_{TM}}\}.$$

- (a) Show that J is not in RE.
 - (b) Show that \overline{J} is not in RE.
 - (c) Show that $J \preceq \overline{J}$.
4. Show that if a language A is in RE and $A \preceq \overline{A}$, then A is recursive.
 5. A language L is **RE-Complete** if:
 - $L \in RE$, and
 - $L' \preceq L$ for all $L' \in RE$.

Recall the following languages:

$$\begin{aligned} L_{\Sigma^*} &= \{\langle M \rangle \mid L(M) = \Sigma^*\} \\ H_P &= \{\langle M, w \rangle \mid M \text{ halts on } w\} \end{aligned}$$

- (a) Is L_{Σ^*} RE-Complete or not? Prove your answer.
 - (b) Is H_P RE-Complete or not? Prove your answer.
6. Show that if A is RE and $A \preceq \overline{A}$, then A is decidable.
 7. Let L_1, L_2 be two decidable languages, and let L be a language such that $L_1 \subseteq L \subseteq L_2$. Is L decidable or not? Prove your answer.
 8. Let L be a language RE. Show that $L' = \{x \mid \exists y : (x, y) \in L\}$ is also RE.
 9. Prove or disprove: there exists an undecidable unary language (a unary language is a subset of 1^*).
 10. **PROBLEM FORMULATION.**
 - (a) Consider the problem of testing whether a TM M on an input w ever attempts to move its head left when its head is on the leftmost tape cell. Formulate this problem as a language and show that it is undecidable.
 - (b) Consider the problem of testing whether a TM M on an input w ever attempts to move its head left at any point during its computation on w . Formulate this problem as a language and show that it is decidable.
 11. Let A and B be two disjoint languages. We say that language C **separates** A and B if $A \subseteq C$ and $B \subseteq \overline{C}$. Show that any two disjoint co-RE languages are separable by some decidable language.

12. Suppose there are four languages A , B , C , and D . Each of the languages may or may not be recursively enumerable. However, we know the following about them:

- There is a reduction from A to B .
- There is a reduction from B to C .
- There is a reduction from D to C .

Below are four statements. Indicate whether each one is

- (a) CERTAIN to be true, regardless of what problems A through D are.
- (b) MAYBE true, depending on what A through D are.
- (c) NEVER true, regardless of what A through D are.

Please, justify your answer!

- (a) A is recursively enumerable but not recursive, and C is recursive.
- (b) A is not recursive, and D is not recursively enumerable.
- (c) If C is recursive, then the complement of D is recursive.
- (d) If C is recursively enumerable, then $B \cap D$ is recursively enumerable.

13. Recall the following definition: A grammar G computes a function f iff for all $u, v \in \Sigma^*$,

$$SuS \Rightarrow_G^* v \text{ iff } f(u) = v.$$

Show a grammar that computes the following functions, defined on unary representations of natural numbers as follows:

- $f_1(n) = 3n + 5$.
- $f_2(n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{3} \\ 11 & \text{if } n \equiv 1 \pmod{3} \\ 111 & \text{if } n \equiv 2 \pmod{3} \end{cases}$
- $f_3(n) = n - 1$.
- $f_4(n) = n/2$. Assume n is even.
- $f_5(w) = ww$, where $w \in \{a, b\}^*$.
- $f_6 = w'$, where $w \in \{a, b\}^*$, and w' is obtained from w by replacing the a 's by b 's and b 's by a 's. For example, $f_6(aba) = bbab$.
- $f_7(a_1 a_2 \dots a_k) = a_1 a_1 a_2 a_2 \dots a_k a_k$, where each a_i is in the alphabet $\{a, b\}$. For example, $f_7(aba) = aaaabbaa$.
- $f_8(w) = \begin{cases} f_6(w) & \text{if the rightmost symbol of } w \text{ is } a \\ f_7(w) & \text{if the rightmost symbol of } w \text{ is } b \end{cases} \quad (\Sigma = \{a, b\})$.

14. Show that the following languages are recursive.

- $L_{29} = \{\langle M \rangle \mid M \text{ is a DFA and } L(M) \text{ is finite}\}$.
- $L_{30} = \{\langle M \rangle \mid M \text{ is a DFA and } L(M) = \Sigma^*\}$.
- $L_{32} = \{\langle M, x \rangle \mid M \text{ is a DFA and } M \text{ accepts } x\}$.
- $L_{33} = \{\langle M, x \rangle \mid M \text{ is a DFA and } M \text{ halts on } x\}$.
- $L_{34} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$.
- $L_{35} = \{\langle M \rangle \mid M \text{ is a DFA and } M \text{ accepts some string of the form } ww^R \text{ for some } w \in \{a, b\}^*\}$.

15. Prove that each of the following languages are not context-free, and write unrestricted grammars that generate them.

- $L_{36} = \{x\#w \mid x, w \in \{a, b\}^* \text{ and } x \text{ is a substring of } w\}$.
- $L_{37} = \{w \in \{a, b, c\}^* \mid \#_a(w) \geq \#_b(w) \geq \#_c(w)\}$.
- $L_{38} = \{a^n b^n c a^n b^n \mid n > 0\}$.
- $L_{39} = \{a^n b^{2n} c^{3n} \mid n \geq 0\}$.
- $L_{40} = \{a^n b^{n+m} c^m d^n \mid m, n \geq 0\}$.
- $L_{41} = \{w \in \{1\}^* \mid w \text{ is the unary encoding of } 2^k \text{ for some } k > 0\}$.

16. Let L_{42} be the language containing only the single string s , where

$$s = \begin{cases} 0 & \text{if } \textit{God does not exist} \\ 1 & \text{if } \textit{God exists} \end{cases}$$

Is L_{42} decidable? Why or why not? (Note that the answer does not depend on your religious convictions.)