Linear Programming

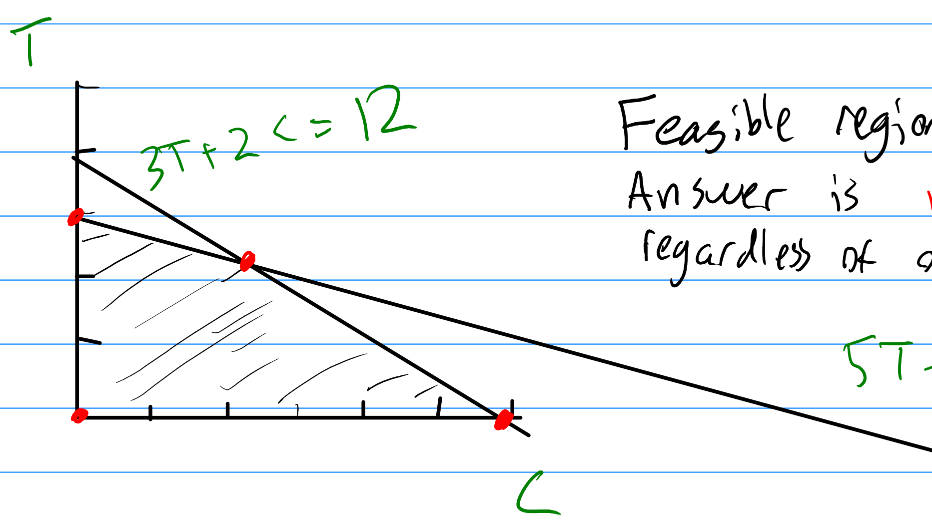
General way to write lots of problems
+ general solver

Developed 1939 by Kantorovich in Stalinist Russia
So think central planning of factories:

- Can produce cars, trucks
- Cars take 2 metal, 1 wood
- Trucks take 3 metal, 5 wood
- Trucks carry twice as much as cars
- You have 12 metal, 15 wood / week

Q: how many cars vs trucks should you produce?

$$\max \quad 2T + C$$
s.t.
$$3T + 2C \le 12$$
$$5T + C \le 15$$
$$T, C \ge 0$$



Feasible region: possible (T,C)
Answer is vertex of it
regardless of objective (2T+C)

Solving by hand Algebraically

2 variables, 4 constraints

⇓

each vertex lies at intersection of 2 constraints

$\binom{4}{2} = 6$ options

$T = 0$: $C = 6$ or $\cancel{C = 15}$ or $C = 0$

infeasible: $(0, 15)$ violates $3T + 2C \le 12$

$C = 0$: $\cancel{T = 4}$ or $T = 3$

Last: $3T + 2C = 12$ & $5T + C = 15$

$\Rightarrow 7T = 18$

$T = \frac{18}{7}$, $C = \frac{15}{7}$

Each give a value $2T + C$:

$(0, 0) \Rightarrow 0$

$(0, 6) \Rightarrow 6$

$(3, 0) \Rightarrow 6$

$(\frac{18}{7}, \frac{15}{7}) \Rightarrow \frac{51}{7}$ largest

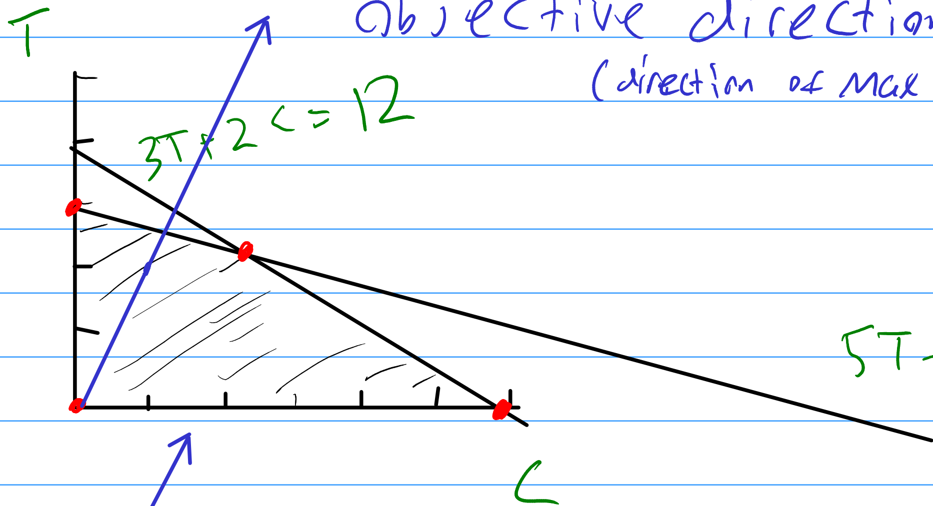
So $(\frac{18}{7}, \frac{15}{7})$ is optimal solution.

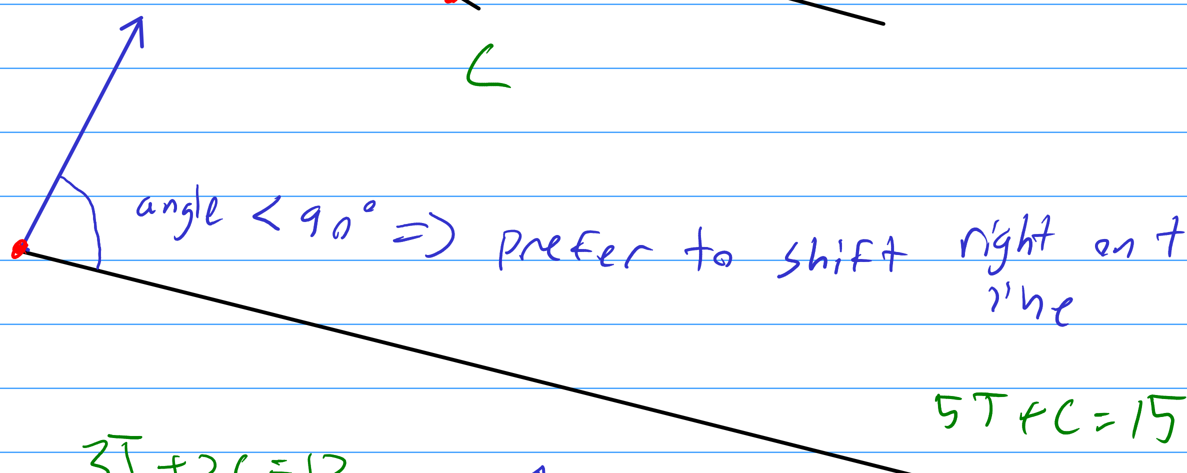
Solving by Hand Geometrically

(challenging w/ >2 variables)

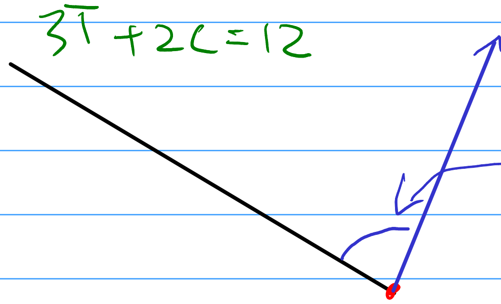
objective direction $2T+C$

(direction of max increase)

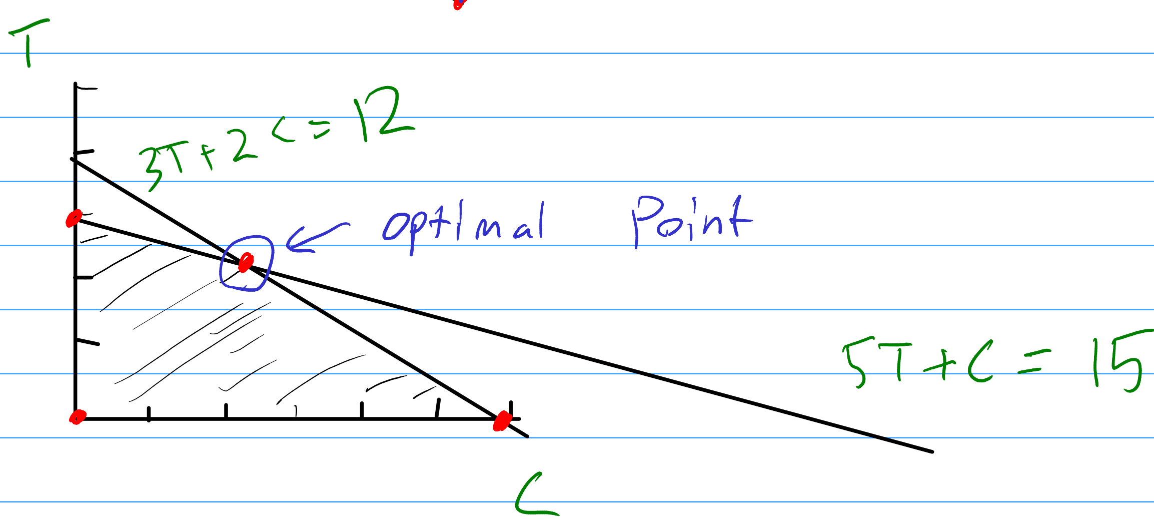




angle $< 90°$ $\Rightarrow$ prefer to shift right on this line



angle $< 90°$ $\Rightarrow$ prefer to shift left



General Linear Programming:

Optimize (max or min) linear objective
subject to linear constraints $(=, \leq, \geq)$

$$\max\ 5x_1 + 6x_2 - 3x_3 + x_4$$

s.t.

$$x_3 \geq x_1 + 2x_2 + 3$$
$$x_4 = x_1 + x_2$$
$$x_1 \geq 0$$
$$x_2 \leq 1$$

Standard Form or "Symmetric"

$$\begin{aligned} \text{Max} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$$\boxed{A}\, x \leq \boxed{b}$$

Alternative Form: No $x \geq 0$ constraint

Equational Form:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Claim can transform between forms without changing problem size much

Standard → Alternative:

$$\begin{bmatrix} A \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ 0 \end{bmatrix}$$

Alternative → Standard:

$x = x' - x''$ where $x', x'' \geq 0$

$$Ax \leq b \iff \begin{bmatrix} A & -A \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} \leq b$$

Standard → equational

"slack" variables s, 1 per eqn in A

$$s = b - Ax$$

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = b$$

$$x, s \geq 0$$

equational → standard

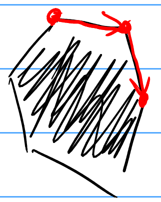
$Ax = b,\ x \geq 0 \iff x \geq 0,\ Ax \leq b,\ -Ax \leq -b$

$$\begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}$$

Algorithms to solve LPs:

Simplex Algorithm

walk along vertices



edge from vertex
corresponds to
equation to relax (until another constraint hit)

[vertex = d constraints = 0 degrees of freedom
edge = $d-1$ constraints = 1 degree of freedom]

Which of the d constraints should you relax each step?

- Many choices, different simplex algorithms
- Standard: direction of steepest increase in objective

Problem: Not polynomial time
(but often fast in practice)

Ellipsoid:
Interior Point
fancier, polynomial time
$O(n^4 \cdot L)$ for L bits of precision

Open Q: strongly polynomial time

Simplex works eventually once you have a feasible vertex.

But: how do you get started?

Problem: if you can solve "does this polytope have any solution" you can also solve LP (= optimize over polytope)

Pf: to tell if $\begin{pmatrix} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{pmatrix} \geq \gamma$, see if $\exists$ any solution to

$$Ax \leq b$$
$$c^T x \geq \gamma$$

$\Rightarrow$ Could tell if answer $\geq \gamma$

$\Rightarrow$ Could binary search on answer.

Solution Find initial vertex by solving different LP w/ simple initial vertex

Want to solve max $c^T x$
s.t. $Ax \le b$, $x \ge 0$

Need to find any x s.t.
$Ax \le b$, $x \ge 0$. ☆

Introduce new variable $z \in \mathbb{R}$
Solve

$$\min z$$
$$Ax - z \le b \qquad ☆☆$$
$$x \ge 0, \quad z \ge 0$$

☆ is feasible $\Leftrightarrow$ ☆☆ has $z=0$ solution

☆☆ has a trivial vertex:

$x = (0, 0, 0, \ldots, 0)$

$z = \max(0, -b_1, -b_2, \ldots, -b_n)$

[why? feasible & (# tight constraints) $\ge$ (# variables)
- all $x_i \ge 0$ constraints tight,
- z constrained by some equation]

$\Rightarrow$ Simplex can get started on (☆☆) and find solution

$\Rightarrow$ Simplex can use (☆☆) soln to start on (☆) & solve it

Duality

let's return to car/truck example.

$P :=$ max $\quad 2T + C$

s.t.

$$3T + 2C \le 12$$
$$5T + C \le 15$$
$$T, C \ge 0$$

Suppose we do not fully solve it
can we at least show the answer
P is not too large?

e.g. $\quad P \le 1000000$?

YES: eqn 2 says $5T + C \le 15$
since $T \ge 0$, $\quad 2T + C \le 5T + C \le 15$.

Or, eqn 1 & $T, C \ge 0$ say: $2T + C \le 3T + 2C \le 12$

Or: eqn 1 + eqn 2 say:
$$8T + 3C \le 27$$
$$\Rightarrow \quad 2T + C = \tfrac{1}{3}(6T + 3C) \le \tfrac{1}{3}(8T + 3C) \le 9.$$

Can we find the best such bound?

$$\alpha \cdot (\text{first eqn}) + \beta \cdot (\text{second eqn})$$
gives value $\quad 12\alpha + 15\beta$
& works if: $\quad \alpha, \beta \ge 0$ $\quad$ [can't subtract inequalities]
$3\alpha + 5\beta \ge 2$ [result $\ge 2T$], $\quad 2\alpha + \beta \ge 1$ [$\ge C$]

Primal | Dual

$P :=$ max $2T + C$
s.t.
$$3T + 2C \leq 12$$
$$5T + C \leq 15$$
$$T, C \geq 0$$

$D :=$ min $12\alpha + 15\beta$
s.t.
$$3\alpha + 5\beta \geq 2$$
$$2\alpha + \beta \geq 1$$
$$\alpha, \beta \geq 0$$

Variables ⟷ Constraints
constraints ⟷ Variables
Objective coefficients ⟷ Constraint Coefficients
max ⟷ min

In general:

standard form

max $c^T x$
s.t. $Ax \leq b$
$x \geq 0$

min $b^T y$
$A^T y \geq c$
$y \geq 0$

nonnegative vars ⟷ inequality constraints

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asymmetric

max $c^T x$
s.t. $Ax \leq b$

min $b^T y$
$A^T y = c$
$y \geq 0$

unconstrained vars ⟷ equality constraints
(why? if T can be negative, don't know $5T \geq 2T$)

Duality

generalization of min cut/max flow

"weak duality": $P \le D$

or: any feasible primal value $\le$ any dual value

i.e. any flow $\le$ any cut

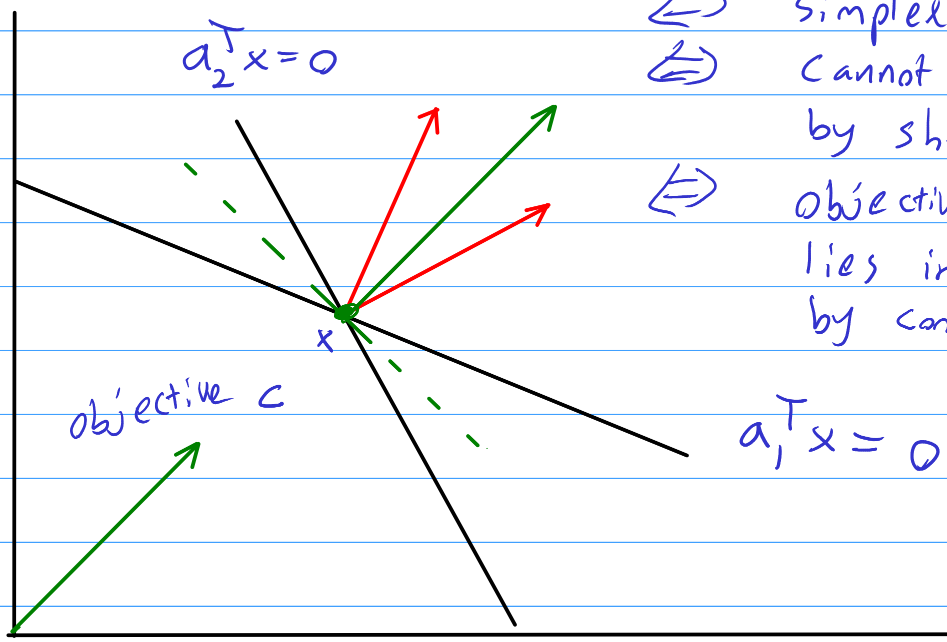
directly follows from construction of D.

"strong duality": $P = D$

best primal $=$ best dual

max flow $=$ min cut

Proof of Strong Duality



x optimal
$\iff$ simplex stops at x
$\iff$ cannot improve objective by shifting x locally
$\iff$ objective direction $\nearrow$ lies in "cone" formed by constraint vectors a_i

$$\boxed{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

Asymmetric version:

$$P: \quad \max\ c^T x \quad | \quad Ax \le b$$
$$D: \quad \min\ b^T y \quad | \quad A^T y = c,\ y \ge 0$$

Let x^* be optimal vertex for P.
I be set of tight constraints

x^* optimal $\Rightarrow$ constraints stop any movement in c direction, or even within 90^0
$\Rightarrow$ c lies in cone of constraints:

$$c = \sum_{i \in I} y_i \cdot a_i \quad \text{for} \quad y_i \ge 0 \ \ \forall i$$

$$c = A^T y, \ \ y \ge 0, \qquad \text{supp}(y) \subseteq I$$

Then

$$b^T y = \sum_{i \in I} b_i y_i = \sum_{i \in I} (a_i^T x^*) y_i$$

I: tight constraints at x^*

$$= \Big(\sum_{i \in I} y_i a_i\Big)^T x^* = c^T x^*.$$

[written out: $\displaystyle = \sum_{i \in I} \Big(\sum_{j=1}^{m} a_{ij} x_j^*\Big) y_i$

$\displaystyle = \sum_{j=1}^{m} x_j^* \Big(\sum_{i \in I} a_{ij} y_i\Big)$

$\displaystyle = \sum_{j=1}^{m} x_j^* c_j = c^T x^*$]

$\Rightarrow$ found pair with $P = D$. ∎