CS 388R: Randomized Algorithms	Fall 2017
Lecture 22 — November 21, 2017	
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1 Overview

This lecture continues our discussion on Markov chains; specifically, chains where

$$P_{uv} = \begin{cases} \frac{1}{d(u)} & (u, v) \in E \\ 0 & \text{Otherwise.} \end{cases}$$

for all edges (u, v). That is, we cover random walks on graphs.

2 Definitions

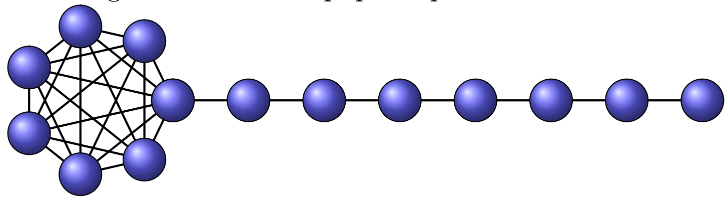
Here we recall a few definitions from last time.

- The hitting time $h_{u,v}$ is the expected number of steps to go from vertex u to a vertex v.
- The commute time $C_{u,v} = h_{u,v} + h_{v,u}$ is the expected number of steps to return to a vertex u after hitting vertex v.
- $C_u(G)$ is the expected time to tour the entire graph G starting at u.
- $C(G) = \max_u C_u(G)$ is the graph's cover time.

In this lecture we are interested in proving some bounds about these properties.

3 Motivating Example: The Lollipop Graph

Figure 1: The Lollipop Graph on 14 Vertices



The lollipop graph on n vertices is a clique of $\frac{n}{2}$ vertices connected to a path of $\frac{n}{2}$ vertices. Let u be any vertex in the clique that does not neighbour a vertex in the path, and v be the vertex at the end of the path that does not neighbour the clique. Then $h_{u,v} = \theta(n^3)$ while $h_{v,u} = \theta(n^2)$. This is because it takes $\theta(n)$ time to go from one vertex in the clique to another, and $\theta(n^2)$ time to successfully proceed up the path, but when travelling from u to v the walk will fall back into the clique $\theta(1)$ times as often as it makes it a step along the path to the right, adding an extra factor of n to the hitting time.

We now wish to prove this formally.

4 Electrical Resistance and Commute Time of a Graph

View graph G as an electrical network with unit resistors as edges. Let $R_{u,v}$ be the effective resistance between vertices u and v. Then the commute time between u and v in a graph is related to $R_{u,v}$ by

$$C_{u,v} = 2mR_{u,v}$$

We get the following inequalities assuming this relation.

If $(u, v) \in E$,

$$R_{u,v} \leq 1 \therefore C_{u,v} \leq 2m$$

In general, $\forall\ u, v \in V$,

$$R_{u,v} \leq n - 1 \therefore C_{u,v} \leq 2m(n - 1) < n^3$$

Here is the high level idea behind the proof: We inject $d(v)$ amperes of current into all vertices $v \in V$. Now fix some vertex $u \in V$ and remove $2m$ current from u leaving net $d(u) - 2m$ current at u. Now we get voltages $x_v\ \forall v \in V$. We will show that $x_v - x_u = h_{v,u}\ \forall v \neq u \in V$ which will give us a relation between commute time and resistance.

4.1 Lollipop Graph

Let us revisit the lollipop graph with the electrical network view and compute $h_{u,v}$ and $h_{v,u}$ with u and v as before. To compute $h_{u,v}$. Let u' be the vertex common to the clique and the path. Clearly, the path has resistance $\theta(n)$. $\theta(n)$ current is injected in the path and $\theta(n^2)$ current is injected in the clique.

Consider draining current from v. The current in the path is $\theta(n^2)$ as $2m - 1 = \theta(n^2)$ current is drained from v which enters v through the path implying $x'_u - x_v = \theta(n^3)$ using Ohm's law ($V = IR$). Now consider draining current from u instead. The current in the path is now $\theta(n)$ implying $x_v - x'_u = \theta(n^2)$ by the same argument.

Since the effective resistance between any edge in the clique is less than 1 and $\theta(n^2)$ current is injected, there can be only $\theta(n^2)$ voltage gap between any 2 vertices in the clique. We get $h_{u,v} = x_u - x_v = \theta(n^3)$ in the former case and $h_{v,u} = x_v - x_u = \theta(n^2)$ in the latter.

4.2 Proof of Relation

Recall that the Laplacian of G is defined by:

$$L_{uv} = \begin{cases} d(u) & v = u \\ -1 & v \neq u \end{cases}$$

and the degree matrix D is:

$$D_{uv} = \begin{cases} d(u) & v = u \\ 0 & v \neq u \end{cases}$$

Consider adding $d(v)$ amps of current to every vertex v, and then removing $2m$ amps from a vertex u. Let x be the voltage vector for the resulting graph. Then we have

$$Lx = i_u = D - 2m\mathbb{1}_u$$
$$\forall v \in V, \sum_{(u,v)\in E} x_v - x_u = d(v) \tag{1}$$

Define $h_{v,u} = 0$ when $v = u$. We can then write

$$h_{v,u} = 1 + \sum_{(v,w)\in E} \frac{1}{d(v)} h_{w,u}$$

because we take one step in our random walk out of v to another vertex $h_{w,u}$ with probability $\frac{1}{d(v)}$ and then have $h_{w,u}$ expected time to reach u. Multiplying through by $d(v)$, we get:

$$d(v) = \sum_{(v,w)\in E} h_{v,u} - h_{w,u} \tag{2}$$

Equations 1 and 2 are linear systems with unique solutions and are identical under $x_v - x_u = h_{v,u}$ (up to same additive shift to each entry). $x_v = h_{v,u}$ if $x_u = 0$.

We have shown that for $i_u = D - 2m\mathbb{1}_u$ with $x = L^+ i_u$ that $x_v - x_u = h_{v,u}$. For u', we have $x' = L^+ i_{u'}$. Now, we have,

$$x - x' = L^+(i_u - i_{u'}) = 2mL^+(e_{u'} - e_u)$$

where e_v is 1 at the entry corresponding to v and 0 elsewhere. The above is equivalent to $2m$ times voltage obtained if you inject 1 ampere at u' and remove 1 ampere from u. Using Kirchoff's law and our previously proven equality that $x_v - x_u = h_{v,u}$ we get

$$\begin{aligned} 2mR_{u,u'} &= (x - x')_{u'} - (x - x')_u \\ &= (x_{u'} - x_u) - (x'_u - x'_{u'}) \\ &= h_{u',u} + h_{u,u'} = C_{u,u'} \end{aligned}$$

5 Cover Time of a Graph

We define $C_u(G)$ as the expected time for a random walk starting at u to visit all vertices in a graph. $C(G)$ is the maximum of $C_u(G)$ over all $u \in V$.

5.1 Bound for $C(G)$

We have $\forall u \in V$,

$$C_u(G) \leq 2m(n-1)$$

Consider the spanning tree T of graph G. The cover time is bounded by traversing the edges of the tree in both directions (as we could just do a DFS on the spanning tree), and hitting time gives the expected time of moving along an edge, we get

$$\begin{aligned} C_u(G) &\leq \sum_{(u,v)\in E(T)} h_{u,v} + h_{v,u} \\ &= \sum_{(u,v)\in E(T)} C_{u,v} \\ &\leq (n-1)\max_u C_{u,v} \\ &\leq 2m(n-1) \end{aligned}$$

This above inequality is tight for lollipop ($\theta(n^3)$) but not for cliques which has $O(n \log n)$ as we can model it as a coupon collector problem.

5.2 Using resistance for a better bound

Let $R_{max} = \max_{u,v \in V} R^{eff}_{u,v}$. Then:

$$mR_{max} \leq C(G) \lesssim mR_{max} \log n$$

Let (u, v) have $R^{eff}_{u,v} = R_{max}$

$$C(G) \geq \max(h_{uv}, h_{vu}) \geq \frac{h_{uv} + h_{vu}}{2} = \frac{C_{uv}}{2} = M \cdot R^{eff}_{uv} = M \cdot R_{max}$$

5.3 Expected time from node u to any node v

$$h_{uv} \leq C_{uv} = 2M \cdot R^{eff}_{uv} \leq 2M \cdot R_{max}$$

So after $8M \cdot R_{max}$ steps, you will be reached v $w.p. 3/4$

Let's repeat the process $log(n)$ times,

$$Pr(never\ reach\ v) \leq (\frac{1}{4})^{log(n)} = \frac{1}{n^2}$$

$$Pr(any\ vnotreached) \leq \frac{1}{n^2} \cdot n = \frac{1}{n}$$

$$\begin{aligned}E[T] &\leq Pr(T \leq B) \cdot B + Pr(T \geq B) \cdot E[T|T \geq B] \\ &\leq 8M \cdot R_{max}log(n) + \frac{1}{n^2} \cdot (2Mn + MR_{max}log(n)) \\ &= \Theta(MR_{max}log(n))\end{aligned}$$

References

[MR] Rajeev Motwani, Prabhakar Raghavan Randomized Algorithms. *Cambridge University Press*, 0-521-47465-5, 1995.