Lecture 8 — September 26, 2017

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NOTE: THESE NOTES HAVE NOT BEEN EDITED OR CHECKED FOR CORRECTNESS

1 Perfect Hashing

1.1 Overview

In the previous lecture, we analyzed Cuckoo Hashing, which still has a very low chance of collision. Also, Cuckoo Hashing requires fully randomize functions. Today we will try to find a perfect hashing scheme with no collisions, and which only requires pairwise randomized.

1.2 Goal

Definition 1. *A hash function h is **perfect** for $S \in [U]$ if it has no collisions, i.e. $h(x) \neq h(y)$ for all $x \neq y$ and $x, y \in S$.*

Goal: For a given set S of size k, find a perfect hash function $h : [U] \to [m]$, we want m as small as possible.

1.3 Intuition and Easy solutions

- Identity function: $h(x) = x$, but with $m = U$.
- Giant lookup table via hash table, actually depends on which hash table you pick.
- Pairwise independent hash function with $m = O(k)$, but not perfect with $O(\frac{\log k}{\log \log k})$ worst case lookup.

Now we still use a pairwise independent hash function h but with more space than $O(k)$, we want to find an upper bound for m such that no collisions will occur.

Lemma 2. *With probability more than $\frac{1}{2}$ we can find a perfect random pairwise independent hash function h with $m = k^2$ for S.*

Proof. Using Markov, we have,

$\Pr[h \text{ is not perfect for } S] = \Pr[h \text{ has at least 1 collision for } D] \leq \mathrm{E}[\text{number of collisions for } h]$.

By expanding E[number of collisions for h] as pairs of collision indicators we get:

$$E = \sum_{x_1 < x_2 \; x_1, x_2 \in S} Pr[h(x_1) = h(x_2)] \leq \binom{k}{2} \max_{x_1, x_2} Pr[h(x_1) = h(x_2)] \leq \frac{k^2}{2m} \tag{1}$$

Therefore we know if we let $m = k^2$ and we randomly choose a hash function that is pairwise independent, we will have failure probability at most a half. □

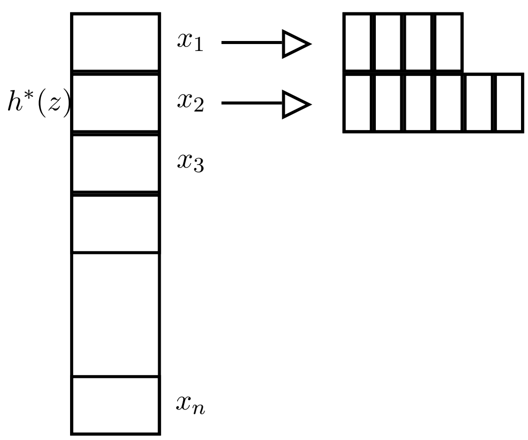
If we design a Las Vegas algorithm to repeatedly find a pairwise independent hash function, after a constant number of tries, we will get a perfect hash function with high probability.

1.4 Perfect Hashing

Now we have a way to find a perfect hash function with $m = k^2$. However, we want $m = O(k)$.

Suppose we first get a random hash function $h^*[U] \to [m]$, with $m = O(k)$. This hash function may have collisions. Create a linked list for each collision.

Now suppose we map each linked list with size k' with a perfect hash function with size k'^2, we can then flatten that link list out and store with some extra space to make it a perfect hash function.



More formally, we have $h^* : [U] \to [m]$, and $h_i : [U] \to [Z_i^2]$ to be a perfect hashing, where Z_i is the number of elements that hash to cell i, or mathematically denoted as $|(h^*)^{-1} \cap S|$.

Record $Y_i = \sum_{j<i} Z_i^2$. We set our final perfect mapping as

$$h(u) = Y_i + h_i(u), where \; i = h^*(u)$$

.

It is then easy to see that h is perfect with range $\sum_{i=1}^{m} Z_i^2$, we need to estimate $\sum_{i=1}^{m} Z_i^2$.

Since total number of collisions equals to

$$\sum_{i=1}^{m} \binom{Z_i}{2} = \frac{1}{2}(\sum_{i=1}^{m} Z_i^2) - \frac{k}{2},$$

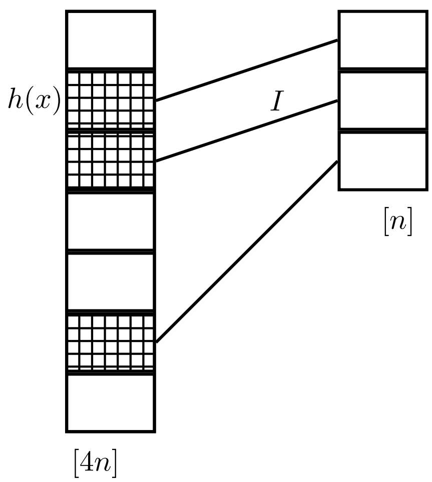
By taking expected value of both sides we have:

$$E[\sum_{i=1}^{m} Z_i^2] = 2E[\text{total number of collisions}] + k \leq \frac{2k^2}{2m} + k \quad (2)$$

If we let $m = k$, we will get $2k$ to be the expected size of our hash function h.

Now we get a Las Vegas algorithm to rebuild h until size of h is less or equal to $4k$. By Markov, each round our success probability is greater than $\frac{2k}{4k} = \frac{1}{2}$, thus with $O(1)$ rounds or $O(k)$ total times we will success with high probability.

If we want to further achieve $m = k$, we can use a lookup table for the perfect hashing, where we index each non-empty element in the mapping of perfect hashing h as I, then we take $h'(x) = I_{h(x)}$ which still runs in $O(k)$ times.



2 Lower bound on hashing

To hash a set S of size k in $[U]$, lots of scheme give $O(k)$ word of space, where 1 word $= \log U$ bits. A natural question is to ask, can we do better?

Suppose the hash table was stored using b bits, then the total number of possible representations you can have is at most 2^b. Since your representations must include all possible subsets of size k of U we see that $2^b \geq (|U|_k) \geq \left(\frac{|U|}{k}\right)^k = 2^{k \log(|U|/k)}$. If $k < \sqrt{|U|}$ then we see $b \geq \frac{1}{2} k \log(|U|)$. However, $\log(|U|)$ is the size of any word, and so we need $\Omega(k)$ words. This means if we need to be able to has ALL POSSIBLE sets, then we cannot do better than a regular hash function.

3 Bloom Filters

This is a set membership data structure with some chance of false positives. In particular, for a particular set S you can get queries of the kind $x \in S$?, if the answer is 'yes' you would like to be always right, however if the answer is 'no', then you are allowed to fail with probability $1 - \delta$. It is possible to do this with $O(k \log(\frac{1}{\delta}))$ bits.

Applications of this structure:

- Use the filter before a slow operation (for example, chrome uses this to maintain a list of malicious urls).
- Database joins ('Does this key have a different entry in the corresponding table?')
- Bitcoin (to speed up wallet synchronization).

Let n be the number of items, m be the number of buckets. The datastructure picks up k uniform random hash functions $h_1, \ldots, h_k$ where k is a parameter to be decided later. You then store $\vec{y} \in \{0,1\}^m$ where $y_j = 1$ iff $\exists x \in S, i \in [k].h_i(x) = j$. Respond with 'yes' to a query on x iff $x \in \cap_{i \in [k]} Y_{h_i(x)}$.

We now analyze the failure probability of this. Let $p =$ the fraction of 0's in an array. $E[p] = Pr[\text{any single entry is 0}] = \left(1 - \frac{1}{m}\right)^{nk} \approx e^{-nk/m}$. The variables negatively associate and hence concentrate, which means p is most probably going to be the expectation, upto a constant. The probability that one of these was 1 is $(1-p)$ and so we have $\delta = (1-p)^k \approx (1 - e^{-nk/m})^k$. We will try to find the k that minimizes this value. To do this, observe that $(1-e^{-nk/m})^k = [(1-e^{-z})^z]^{n/m}$ where $z = \frac{nk}{m}$. It sufficies to minimize with respect to z which can be done by differentiating the log and setting it to 0. It turns out that at the minimum $k = \frac{m}{n}\ln(2)$ and $\delta < \frac{1}{2^k} = 0.618^{\frac{m}{n}}$. Setting $m = O(n \log(1/\delta))$ does the job.