# Problem Set 6

Sublinear Algorithms

Due Tuesday, October 20

1. In this problem we show that matrices that satisfy the RIP-2 cannot be very sparse. Let $A \in \mathbb{R}^{m \times n}$ satisfy the $(k, 1/2)$ RIP for $m < n$. Suppose that the average column sparsity of $A$ is $d$, i.e., $A$ has $nd$ nonzero entries.

   Furthermore, suppose that $A \in \{0, \pm\alpha\}^{m \times n}$ for some parameter $\alpha$.

   (a) By looking at the sparsest column, give a bound for $\alpha$ in terms of $d$.

   (b) By looking at the densest row, give a bound for $\alpha$ in terms of $n, m, d$ and $k$.

   (c) Conclude that either $d \gtrsim k$ or $m \gtrsim n$. (Recall that this means: there exists a constant $C$ for which $d \geq k/C$.)

2. Let $T$ to be a 1/4-cover of the unit $\ell_2$ ball $\mathcal{B} \subset \mathbb{R}^n$, meaning that $T \subset \mathcal{B}$ and, for every $x \in \mathcal{B}$, there exists an $x'$ in $T$ such that $\|x' - x\|_2 \leq \frac{1}{4}$.

   (a) Show that it is possible to pick such a $T$ with $|T| \leq 9^n$.

   (b) Let $M \in \mathbb{R}^{n \times n}$ be a real matrix. Show that, for any $y \in \mathbb{R}^n$ with $\|y\| \leq 1$,

   $$|y^T M y| \leq 2 \cdot \max_{x_1, x_2 \in T} |x_1^\top M x_2|$$

   (c) Prove the following lemma:

   **Lemma.** *There exists a set $T' \subset \mathbb{R}^n$ of $2^{O(n)}$ unit vectors such that, for any symmetric matrix $M \in \mathbb{R}^{n \times n}$,*

   $$\|M\| \leq 4 \max_{x \in T'} |x^\top M x|.$$

   **Hint:** Yrg G' vapyhqr gur fhz, naq qvssrerapr, bs nal gjb irpgbef va G (fpnyrq qbja gb yvr ba gur onyy).

   (d) Show that $A \in \mathbb{R}^{m \times n}$ with i.i.d. $N(0, 1/m)$ entries with high probability satisfies the $(k, \epsilon)$-RIP with $m = O(\frac{1}{\epsilon^2} k \log \frac{n}{k})$. [For comparison, in class we only proved $O(\frac{1}{\epsilon^2} k \frac{n}{\epsilon k})$.] **Hint:** Hfr gur fcrpgeny punenpgrevmngvba bs EVC, cvpx n qvssrerag Z sbe rnpu fvmr-x fhccbeg, naq fubj gur fcrpgeny abez bs rnpu fhpu Z vf ng zbfg rcf.

3. Final project. Brainstorm ideas for what your final project will be on, and write a paragraph describing what you might do and who you might work with.