Problem Set 7

Sublinear Algorithms

Due Thursday, October 29

1. In this problem, we show a lower bound for randomized ℓ_2/ℓ_2 *adaptive* compressed sensing with *Fourier* measurements. Let $F \in \mathbb{C}^{n \times n}$ be a fixed orthogonal matrix with $|F_{i,j}| = 1$ for all i, j. [Examples include the Fourier matrix, where $F_{i,j} = e^{2\pi i j \sqrt{-1}/n}$, or the Hadamard matrix.]

   An *adaptive recovery algorithm* with measurements from F will, for an unknown approximately sparse vector x, repeatedly choose an index i_t and see $z_t = (Fx)_{i_t}$. Each choice of index may depend on previous observations. After m observations, the algorithm should return a vector $\widehat{x}$ with

   $$\|\widehat{x} - x\|_2 \leq C \min_{k\text{-sparse } x_k} \|x - x_k\|_2$$

   with 3/4 probability, for some constant C. We will show that such an algorithm must have $m = \Omega(k \log(n/k)/\log\log n)$.

   (a) Show that there exists a set $S \subset \mathbb{R}^n$ of k-sparse vectors such that (I) $\|x\|_2 = \sqrt{k}$ for all $x \in S$, (II) $\|x - y\|_2 \geq \sqrt{k}$ for all $x \neq y$, $x, y \in S$, (III) $\|Fx\|_\infty \lesssim \sqrt{k \log n}$ for all $x \in S$, and (IV) $\log |S| \gtrsim k \log(n/k)$. (Hint: Recall the large set $S \subset \{0,1\}^n$ of k-sparse vectors from class, and flip the signs randomly.)

   (b) Consider choosing a random $x \in S$ and $w \sim N(0, \sigma^2 I_n)$ for $\sigma < \frac{1}{100C}\sqrt{k/n}$, and setting $x' = x + w$. Show that successful sparse recovery $\widehat{x}'$ of x' is sufficient to uniquely identify x.

   (c) Show that this implies $I(\widehat{x}'; x) \gtrsim k \log(n/k)$.

   (d) Separately, use Gaussian channel capacity to show that, for every fixed i,

   $$I((Fx')_i; x) \lesssim \log\log n.$$

   (e) Show that this implies

   $$I((z_1, \ldots, z_m); x) \lesssim m \log\log n.$$

   (f) Conclude that $m \gtrsim k \log(n/k)/\log\log n$.