Homework #1, CS 329E, Spring 2008

Basic information:

- Due date: Jan 31, 2008 (in class)
- You can discuss the problems with other students, but you should write it up yourself. Also, later quizzes are likely to be based upon these (and other homeworks), so please be sure you understand the solutions you write down.
- Please put your name (clearly) on each page of your homework, and staple all the pages together.

Problems:

1. Phrase each of the following real world problems as a graph-theoretic problem:

   (a) You are given a collection of people, and you know which people know each other (which you assume is symmetric). You want to find a group of people who, between them, know everyone. You'd like this group to be as small as possible.

   (b) It is 1900, and you are the match maker for a small village in the Ukraine. Thus, your job is to marry off all the unmarried young men and women. You know which pairs of men and women are willing to be married (and, more importantly, whether their parents are willing to let their children get married, which depends also upon the dowry and other factors). You get a fixed fee for each marriage you make. Under the assumptions that you cannot marry anyone to more than one person and you cannot perform any same sex marriages, you'd like to maximize the fee you'll be paid.

   (c) You are the social organizer for your friends' graduations, and you need to arrange a number of graduation parties. The only problem is that some people can't stand each other, and so you can't have them at the same party. The main expense in each party is renting the nightclub where the party takes place, and so you want to have as few parties as you can arrange.

2. Show how you'd use an oracle for the Yes/No problem to construct a solution to each of the construction problems below.

   (a) Given a graph, determine if it has a clique of size 5, and return it if it does.

(b) Given a set X of items of different weights, and given a specific target total weight B, determine if there is a subset of X of total weight B. (For example, if the input has set $X = \{1, 5, 8, 13, 21, 27, 30\}$ and $B = 41$, then the answer is *yes*, and the subset would be $\{1, 13, 27\}$.)

(c) Given a sequence of integers $a_1, a_2, \ldots, a_n$, find an increasing subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ (so $i_1 < i_2 < \ldots i_k$) of maximum length k. (Thus, the solution to $1, 2, 5, 3, 4, 8, 5, 1, 9$ is $1, 2, 3, 4, 5, 9$, which has length 6.)

(d) (Extra Credit) Given a graph, determine if it has a 3-coloring of the vertices (that is a coloring of the vertices so that no two adjacent vertices have the same color).

3. Express each of the following functions recursively (i.e., give the value for $p(1)$ and define $p(n)$ in terms of $p(n-1)$):

   (a) $p(n) = n^3$

   (b) $p(n) = 2^n$

   (c) $p(n) = n!$

   (d) $p(n) = n^2 + n$

4. Pick any *one* of the functions from the previous problem, and prove that it equals its recursive definition, using induction.

5. For each of the following pairs of functions $f(n)$ and $g(n)$, state (without proof) which of the following cases is true: (a) $f(n)$ is $O(g(n))$, (b) $g(n)$ is $O(f(n))$, (c) each is big-oh of the other (i.e., $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$), or (d) neither is big-oh of the other.

   (a) $f(n) = 3n^2, g(n) = 5n^3$

   (b) $f(n) = \log n, g(n) = 500$

   (c) $f(n) = (\log n)^5, g(n) = n/2,$

   (d) $f(n) = \sqrt{n}, g(n) = \log n,$

   (e) $f(n) = \sqrt{n}, g(n) = n,$

   (f) $f(n) =$

   - 1 if n is odd
   - n if n is even

   $g(n) = n$

6. Prove that $3n^2$ is $O(n^2 + 100)$ (by producing the two constants).