CHALLENGE PROBLEMS IN FIRST-ORDER THEORIES

BENJAMIN SHULTS
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Challenge Problems in First-Order Theories

Benjamin Shults
bshults@math.utexas.edu
The University of Texas at Austin
Austin TX 78712, USA

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1 IPR

IPR is a tableau-base theorem prover which implements rules for handling a knowledge base of axioms, theorems and definitions [6]. It tries to be selective in its choice of theorems to apply and how to apply them. It does this by taking advantage of the representation of the knowledge in the knowledge base and common-sense restrictions to fetching.

When selecting a theorem to apply in a proof, IPR generally follows the following guidelines:

- The theorem should have something to do with the problem at hand.
- The theorem should (generally) not add more things to be proved but preferably finish part of the proof.

IPR handles equality using an incomplete substitution method advocated by Frank Brown [2].

IPR also has a very nice interface. When run interactively, the unused formulas from a branch of the tree are presented in English with the header “Suppose” over the positive formulas and “Show one of the following” over the negative formulas. When a proof is found, the condense algorithm is applied [5] and the proof is output in English.

2 Examples

Here we briefly describe some examples of theorems proved by the IPR system in the presence of relatively large knowledge bases.

In each example, the knowledge base is designed to have the property that for each sequent, there is a sequence of sequents in the knowledge base which
form a chain relating some predicate in the sequent to some predicate in the challenge problem. This ensures that it is possible for even the restricted rule used by IPR to apply each theorem in the knowledge base.

In the first example, the proof itself is very short and easy to find. The difficulty of this problem comes from the fact that IPR found this proof in the presence of a knowledge base of over 100 sequents, each taken from earlier sections of Kelley’s text. In the second and following examples, the proof itself is rather complex.

**Example 1** The challenge is part of the 101st labeled theorem from John Kelley’s *General Topology* [3]. This is Theorem 19 on page 147.

*If a product is locally compact, then each coordinate space is locally compact.*

Which is formalized as

$$(\forall X)(\forall A)(\text{locally-compact}(\prod_{A} X) \supset (\forall a)\text{locally-compact}(X_a))$$

This is true in the following theory:

$$(\forall X)(\forall A)(\forall a)\text{continuous-from-to}(\pi_a, \prod_{A} X, X_a)$$

$$(\forall X)(\forall A)(\forall a)\text{open-from-to}(\pi_a, \prod_{A} X, X_a)$$

$$(\forall f)(\forall A)(\forall B)(\text{(open-from-to}(f, A, B) \wedge \text{continuous-from-to}(f, A, B)\wedge$$

$$\text{locally-compact}(A)) \supset \text{locally-compact}(B))$$

Here, $\prod_{A} X$ represents the product topology where $X$ is a bijection from the index set $A$ to a set of topologies. We use $\prod_{A}$ rather than $\prod A$ to distinguish the topology from the underlying set.

Because this proof is short, we present here the English proof output by IPR. Bound variables are symbols preceded by the underscore character “_”. Skolem constants are also preceded by the underscore character and surrounded by parentheses. If $f$ is a function, then the application of $f$ to $x$ is denoted $\{f\}(x)$. The three theorems used are labeled in the knowledge base by the strings “a statement on page 147 of Kelley,” “Theorem 3.2 in Kelley” and “a statement on page 90 of Kelley.” The rest of the theorems in the knowledge base were taken from the earlier parts of the same text relating the the predicates involved in the statement of the challenge. The complete input is available from the author.

**Theorem:**

If the product of $(_X_)$ over the index set $(_A_)$ is locally compact then for every $(_A_)$ $\{(X_\_)(_A_)$ is locally compact.}
Proof:
Suppose that
the product of \((X_n)\) over the index set \((A_n)\) is locally compact
and show that for every \(A \in \{X_n\}(A)\) is locally compact.
Replace the first bound variable in the formula:
for every \(A \in \{X_n\}(A)\) is locally compact with the new term \((A)\).
Since we know that
the product of \((X_n)\) over the index set \((A_n)\) is locally compact
and we are trying to show that \(\{X_n\}(A)\) is locally compact
we can apply a statement on page 147 of Kelley.
Now we only need to show that
the \((A_n)\)th projection function of \((X_n)\) over \((A_n)\) is an open
function from the product of \((X_n)\) over the index set \((A_n)\) onto
\(\{X_n\}(A)\)
and
the \((A_n)\)th projection function of \((X_n)\) over \((A_n)\) is a continuous
function from the product of \((X_n)\) over the index set \((A_n)\) to
\(\{X_n\}(A)\).

1. Since we are trying to show that
the \((A_n)\)th projection function of \((X_n)\) over \((A_n)\) is an open
function from the product of \((X_n)\) over the index set \((A_n)\) onto
\(\{X_n\}(A)\)
we can apply Theorem 3.2 in Kelley
which finishes that branch of the proof.

2. Since we are trying to show that
the \((A_n)\)th projection function of \((X_n)\) over \((A_n)\) is a continuous
function from the product of \((X_n)\)
over the index set \((A_n)\) to \(\{X_n\}(A)\)
we can apply a statement on page 90 of Kelley
which finishes that branch of the proof.

Example 2 We want to prove

\((\forall S)(\text{Hausdorff}(S) \supset \text{closed-in} (\text{top-to-class} (\text{the-diagonal-of}(S)), S \times r S)).\)

The following seven formulas are all that is needed in the proof. They actually
contain a bit more information than what is needed.

\[(\forall X)(\forall S)(X \in \text{top-to-class}(\text{the-diagonal-of}(S)) \iff (\exists A)(A \in \text{top-to-class}(S) \land X = \{A, A\})\]

\[(\forall A)(\forall B)(\forall C)(\forall D)((A, B) = (C, D) \supset (B = D \land A = C))\]

\[(\forall A)(\forall B)(\text{disjoint}(A, B) \iff \neg(\exists Y)(Y \in A \land Y \in B))\]

\[(\forall X)(\forall S)(\forall T)(X \in S \times T \iff (\exists A)(\exists B)(A \in S \land B \in T \land X = \{A, B\}))\]

\[(\forall X)(\forall S)(\forall T)(X \in \text{top-to-class}(S \times T) \iff (\exists A)(\exists B)(A \in \text{top-to-class}(S) \land B \in \text{top-to-class}(T) \land X = \{A, B\}))\]

\[(\forall X)(\text{Hausdorff}(X) \iff (\forall A)(\forall B)((A \in \text{top-to-class}(X) \land B \in \text{top-to-class}(X) \land A \neq B) \supset (\exists G_1)(\exists G_2)(\text{open-in}(G_1, X) \land \text{open-in}(G_2, X) \land A \in G_1 \land B \in G_2 \land \text{disjoint}(G_1, G_2)))\]

\[(\forall X)(\forall A)(\text{closed-in}(A, X) \iff (\forall y)(y \in \text{top-to-class}(X) \land y \not\in A) \supset (\exists G)(y \in G \land \text{open-in}(G, X) \land \text{disjoint}(G, A)))\]

Notice that \(\times\) is the product topology on the product of topological spaces, whereas \(\times\) is simple cartesian product of sets. Also the-diagonal-of a topological space, \(S\), represents a subspace (rather than a subset) of \(S \times S\).

The proof is found, pared down to its shortest form and printed in English by IPR in about 30 seconds.

**Example 3** Here is an example from the theory of vector spaces [1].

\[(\forall W)(\forall V)((a \text{- vector-subspace}(W, V) \land a \text{- vector-space}(V)) \supset (\exists E)(\exists F)(\text{basis-of}(E \cup F, V) \land \text{basis-of}(E, W)))\]

IPR finds the proof using the knowledge base formed from the following five formulas:

\[(\forall b)(\forall V)(\text{basis-of}(b, V) \supset (\text{lin-ind-subset}(b, V) \land b \subset \text{vec-to-class}(V)))\]

\[(\forall s)(\forall V)(\forall t)(\text{lin-ind-subset}(s, V) \land \text{basis-of}(t, V) \supset (\exists u)(u \subset t \land \text{basis-of}(s \cup u, V)))\]

\[(\forall A)(\forall B)(\text{a-vector-space}(A) \supset (\exists b)(\text{basis-of}(b, A)))\]

\[(\forall A)(\forall B)(\text{a-vector-space}(A, B) \supset \text{a-vector-space}(A))\]

\[(\forall W)(\forall V)(\forall e)((a \text{- vector-subspace}(W, V) \land e \subset \text{vec-to-space}(W)) \supset (\text{lin-ind-subset}(e, W) \iff \text{lin-ind-subset}(e, V)))\]

**Example 4** Here is an example from homotopy theory [4].

\[(\forall X)(\forall x_0)(\forall x_1)((\text{path-connected}(X) \land x_0 \in X \land x_1 \in X) \supset \text{isomorphic-groups}(H_1(X, x_0), H_1(X, x_1)))\]
The knowledge base formed from the following formulas is sufficient although a bit more than necessary for the proof. The proof was found with these theorems (which are more than needed) and others in the knowledge base by IPR.

\[(\forall A)(\forall B)(\text{isomorphic-groups}(A, B) \iff (\exists f)\text{group-isomorphism}(f, A, B))\]

\[(\forall X)(\forall x_0)(\forall x_1)((\text{path-connected}(X) \land x_0 \in X \land x_1 \in X) \supset\]

\[(\exists p)\text{path-from-to}(p, x_0, x_1, X))\]

\[(\forall X)((\forall x_0)(\forall x_1)((x_0 \in X \land x_1 \in X) \supset (\exists p)\text{path-from-to}(p, x_0, x_1, X)) \supset\]

\[\text{path-connected}(X))\]

\[(\forall a)(\forall x_0)(\forall x_1)(\forall X)\text{path-from-to}(a, x_0, x_1, X) \supset\]

\[\text{group-isomorphism}(\tilde{a}, H_1(X, x_0), H_1(X, x_1))\]

The source code for the implementation and the input for these examples are available from the author.

References


