Qualitative and Quantitative Simulation: Bridging the Gap

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Abstract

Shortcomings of qualitative simulation and of quantitative simulation motivate combining them to do simulations exhibiting strengths of both. The resulting class of techniques is called semi-quantitative simulation. One approach to semi-quantitative simulation is to use numeric intervals to represent incomplete quantitative information. In this research we demonstrate semi-quantitative simulation using intervals in an implemented semi-quantitative simulator.

Q3 progressively refines a qualitative simulation, providing increasingly precise predictions which can converge to a numerical simulation if the original qualitative simulation is sufficiently refined. Q3's simulations are based on a technique which guarantees correctness guarantees from qualitative and interval simulators. Q3 is intended for relatively few qualitatively distinct qualitative states, to improve

not constituted
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1 Introduction

Systems that change over time are often so complex that analytical solutions, equations predicting future system states as a function of time, cannot be found. In those cases simulation is useful for prediction. Given a model of system structure and initial state, simulation determines trajectory through its state space.

When accurate numerical information about structure and initial state of numerical simulation techniques is available, even only qualitatively available, a significant body of work describes methods for stronger predictions than quantitative simulation. As an example...
An object is fired upward fast enough to escape a gravitational field

Figure 1: Qualitative simulation of an object fired upward at greater than escape velocity shows that the gravitation experienced by the object produces a negative acceleration (a), reducing velocity (b). As distance increases (c), gravitation decreases. Qualitative simulation is another behavior in which the object falls back to Earth (not shown).

This paper significantly revises and expands a preliminary account and provides a proof of convergence and stability for step size.

2 Q3 and Step Size Refinement

Q3 improves on pure qualitative simulation by augmenting inferences when quantitative information is missing, behaviors, and in providing numerical bounding spaces of possible extracting as much explicitly.
tative simulations because the qualitative features of qualitative simulation trajectories do change significantly from one time point to the next. Therefore step sizes for qualitative simulations are large by definition and so numerical inferences on them tend to be weak.

Augmenting a Q2 simulation so that it has smaller step sizes can lead to quantitative inferences, just as numerical simulations can be improved within limits imposed by the accuracy of floating point arithmetic. Q3 augments Q2 with the the capability of smaller step sizes. Q3 first generates qualitative behavior via Q2. Then, better inference step size refinement and augmentation technique. Adaptively represent a the
Figure 2: A simple tank model and its overflow behavior. The model consists of two constants, shown in (a). These constants hold at all times. Instantiating the template at a potentially distinctive time point TO and T1 leads to representations of its behavioral network representation of the overflow behavior is shown in (b). All time points would be unwieldy to depict graphically, but can be memory. The same behavior represented graphically is a quantity of fluid in the tank. Overflow occurs at the capacity of the tank. The mean value const.

\[
\begin{align*}
\text{Netflow}(t) &= \text{Inflow}(t) - \text{Outflow}(t) \\
\text{Amount}(t) &= \int_{0}^{t} \text{Netflow}(t)
\end{align*}
\]
Figure 3: Constraint propagation of interval labels through an add constraint. The interval at each terminal is narrowed by using the constraint to propagate the intervals currently at the other terminals, so an add constraint actually enforces three relations, one addition, $Z \subseteq X + Y$, and two subtractions, $X \subseteq Z - Y$ and $Y \subseteq Z - X$.

Variables concerned at particular points in time (Figure 2b). These constraints relate in quantitative bound the qualitative landmarks of model variables. The constraint narrowing of one or more intervals (Davis 1987), where a narrower interval affects other about quantitative value. When an interval is narrowed, the constraints directly affected can often narrow other (Figure 13). Thus the effect of narrowing an interval on the constraint network.

There are a number of different kinds:

- Arithmetic and non-arithmetic
- Greatest and less than
- Mean value constraint propagation
Interval propagation across an $M^+$ constraint: $Y = M^+(X)$

1. Project $Y$ across the envelopes.
2. Intersect the projection with the old interval for $X$.

The interval $Y$ implied by $Y$.

The old interval for $X$.

Intersection = new interval for $X$.

Figure 4: Propagation through an $M^+$ (positive monotonic) function with quantitative envelopes.

Hatched regions of the axes represent intervals, and the upper and lower envelopes bound a space of monotonic functions. Propagation shown is from an interval on the $y$-axis to the $x$-axis. Propagation the other way, from $x$ to $y$, is analogous.

If two variables are monotonically related, the highest and lowest points of an interval imply highest and lowest points of the projection of that interval on

A qualitative monotonic function represents a large set of quantitative monotonic functions, the middle ground between qualitative monotonic functions. The upper and lower monotonic envelopes which bound this monotonicity are illustrated by Figure 4, which shows an

2.1.3 The mean value constraint

The mean value constraint is designed to

which states:

where time $t^* \in (t_{n-1}, t_n)$.

(Forrester 1961) relates

We do not know the value

$\text{ret}$

See e.g. Hyvönen (1992 p. 89) techniques (Beltyukov 1964).
because \( t^* \in (t_{n-1}, t_n) \), because a closed interval \( I = [a, b] \) is a superset of the open interval \((a, b)\), and because the qualitative simulation module (QSIM) ensures that \( r(t) \) and all other varying quantities are monotonic between states at adjacent time points \( t_{n-1} \) and \( t_n \).

From (1) and (2),

\[
\frac{\text{lead}(t_n) - \text{lead}(t_{n-1})}{t_n - t_{n-1}} \in \left[ \text{ii}(\text{rtd}(t_{n-1})), \text{rtd}(t_n) \right], \quad \text{ii}(\text{rtd}(t_{n-1})), \text{rtd}(t_n) \right]. \tag{3}
\]

When quantities are known only to within intervals, this equation must be intervalized in the obvious ways. Real variables \( x \) are replaced by corresponding interval variables \( X \), real functions \( f(x) \) are replaced by corresponding interval functions \( F(X) \), real arithmetic operators \( +, -, \cdot, / \) are given the corresponding interval interpretations, the natural interval (Moore 1979) of \( \varepsilon \) is \( \subseteq \) and functions \( \text{ii}(\cdot) \) and \( \text{ui}(\cdot) \) are applied respectively to the bounds of intervals. Intervalizing (3) gives

\[
\frac{LEVI(T_n) - LEVI(T_{n-1})}{T_n - T_{n-1}} \leq \left[ \text{ii}(\text{RAT}(T_{n-1})), \text{RAT}(T_n) \right], \quad \text{ui}(\text{RAT}(T_{n-1})), \text{RAT}(T_n) \right].
\]

where the lower bound of an interval \( X \) is denoted by \( \underline{X} \) and the high bound by \( \overline{X} \) (Moore 1979).

The RHS simplifies to the convex hull of the set \( \text{RAT}(T_{n-1}) \cup \text{RAT}(T_n) \), or \( \text{RAT}(T_{n-1}) \cup \text{RAT}(T_n) \), where the convex hull includes everything in either interval or between them. This results in the

**mean value constraint**:

\[
\frac{\text{RAT}(T_n) - \text{RAT}(T_{n-1})}{T_n - T_{n-1}} \leq \text{RAT}(T_{n-1}) \cup \text{RAT}(T_n). \tag{4}
\]

Equation (4) can be solved algebraically for each variable on the left hand side. The resulting right hand side can then be evaluated to give an interval, which is intersected with the quantity’s current interval as in Figure 3.

Quantitative inferences provided by the mean value constraint tend to be weak when the variables \( T_{n-1} \) and \( T_n \) are widely separated, as is often the case with qualitatively distinct time scales. Better results are typically obtained from the mean value constraint, which makes adjacent time points closer together (as in Figure 6a). The result is an interval constraint which is more sensitive to changes in the quantity.

**Alternatives to the mean value constraint** The mean value constraint is a robust method. An obvious improvement over the relatively weak Euler’s method, a mainstay of numerical simulation, is to use an interval method (such as Runge-Kutta) can be extended to interval methods, such as the Brouwer fixed-point theorem (Theorem 5.7). Moore (1979 p. 94–97) also describes a simulation. Brouwer’s proof of the fixed-point theorem shows that the feasible simulation.

**2.2 Phase II: progress**

A quantitatively annotated model refined in Phase I is an initial guess for the model.
2.2.1 Step size refinement: overview

Standard numerical simulation algorithms estimate system state at the next time point by extrapolating from current trends. It is better to extrapolate only a short distance along the system trajectory and then to reassess current trends before extrapolating further, than to extrapolate a longer distance. This means keeping the step size of the simulation small and, intuitively, small step sizes typically lead to less error in the predicted trajectory of the system.

For most interval generalizations of numerical simulation methods, small step sizes lead to narrower but correct interval predictions. Step size refinement, the step sizes of the simulation algorithm, are sharp predictions in the form of narrow intervals.

These intuitions about step size refinement are proven in Appendix B.

2.2.2 Step size refinement

When are given a finite order ordinary differential equation in the interval range \([t_i, t_{i+1}]\), step size refinement may overlap at iteration.
Figure 5: A rocket is fired upward from the Earth's surface. Gravity decreases with the squared distance from the Earth's center (Figure 8). Initial velocity is $v_0 \in [10000, 20000] \text{ m/s}$, which contains escape velocity (approximately $11000 \text{ m/s}$). Q3 correctly predicts that the rocket could either return to Earth or escape. The return behavior is the one shown, with graphs for height, acceleration vs. time. Inferences about qualitative time points $T_0$, $T_1$, and...
Figure 6: A rocket is fired upward at \([3000, 3300]\) meters per second, less than escape velocity. The behavior in which the rocket falls back to the ground is shown in (a), which also shows weak quantitative inferences that were unable to prune any of the other two behaviors. In (b) the behavior is shown, however 25 time points were interpolated. Consequently, quantitative are much stronger — and the 2 impossible behaviors have been pruned. Table 1 intermediate stages in the simulation.
<table>
<thead>
<tr>
<th>Variable</th>
<th>Acceleration</th>
<th>Velocity</th>
<th>Height</th>
<th>Time</th>
<th># of h's</th>
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<tr>
<td>TIME</td>
<td>153</td>
<td>153</td>
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<td>T1</td>
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<tr>
<td>(a)</td>
<td>[-9.83, 0]</td>
<td>(-\infty, 0]</td>
<td>0, 305</td>
<td>305, 3</td>
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<td>Phase I</td>
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<td>(b)</td>
<td>[-9.16, -9.16, 1496, , 229, 229, 316, 316]</td>
<td>[-8.44, 0]</td>
<td>2009</td>
<td>505, \infty</td>
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<td>1 interp.</td>
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<td>(c)</td>
<td>[-8.99, -8.77, 1535, , 289, 373, 326, 326]</td>
<td>[-8.57, 0]</td>
<td>1948</td>
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<td>[-8.91, -8.65, 1558, , 319, 422, 331, 51]</td>
<td>[-8.64, 0]</td>
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<td>424, \infty</td>
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</table>

Table 1: The returning rocket simulation at various variables are each shown at TIME=153 (the algorithm, and either T1 when the rocket is on the way down. Each technique interval results get narrated in the text.)
Figure 7: \( T_{i-1} \) and \( T_i \) are time points, but their values are known only to within intervals, indicated by solid line segments. Between them is a gap. \( T_{i-1} \) and \( T_i \) are the endpoints of a time size is in \([w(gpp), \in\), where \( w(gpp) \) is the width of the gap. \( S \) is the maximum possible size from \( T_{i-1} \) to \( T_i \).

3 Detailed Example: a Nonlinear Second-Order Rocket

We now step through an example requiring step size refinement and the auxiliary offers. Consider a rocket in a gravitational field which decreases with the second order and nonlinear, and hence makes a useful demonstration model appears in Figure 8. The simulation for this example gives known quantitative data about the Earth's gravity greater than the escape velocity of 11,000 m/s so that simulation is useful.

To direct Q3's operation, we specify:

Goal: “Preserve as many behavior(s).”

To minimize the potential cost, we select a subgoal to pursue, giving

Subgoal 1:

Phase 1 of a phase 2 of the...
(define QDE escape-velocity
  (text "Gravity decreases with height as \( r'' = -GM/r^2 \)"
  ; Define model variables and their qualitative values.
  (quantity-spaces
    (r  (0  sea-level inf)  "meters from Earth's core")
    (r^2 (0  inf  )  "distance squared"  )
    (h  (0  inf  )  "meters above surface"  )
    (km (0  inf  )  "kilometers above surface")
    (surface (0  s*  )  "depth of Earth"  )
    (dr/dt (minf 0  r*  inf)  "velocity, m/s"  )
    (d2r/dt2 (minf 0  inf)  "acceleration, m/s^2"  )
    (G (0  G*  )  "Gravitational constant G")
    (Earth-M (0  M*  )  "Mass of Earth M"  )
    (K (0  K*  )  "K (=G*M)"  )
    (-K (-K* 0  )  "-K"  )
    (=1000 (0 1000  )  "Thousand"  )
  )

; The model defines these constraint templates
  (constraints
    ((mult  km  =1000  h )  )
    ((add  surface  h  r  )  (s* 0  sea-level))
    ((mult  r  r  r^2  )  )
    ((mult  G  Earth-M  K  )  (g*  m*  k*)  )
    ((minus  k  -k  )  (k*  -k*)  )
    ((mult  d2r/dt2  r^2  -K  )  )
    ((d/dt  r  dr/dt  )  )
    ((d/dt  dr/dt  d2r/dt2  )  )
    ((constant  surface)  )
    ((constant  G))
    ((constant  Earth-M))
    ((constant  k))
    ((constant  -k))
    ((constant  =1000))))

Figure 8: Qualitative model of a second order nonlinear system. A list of quantity-spaces describes the qualitative values the various model variables can have, and a list of constraints describes the relationships among those model variables. This model describes an object in free fall in the Earth's gravitational field, such as a rocket or other projectile fired upward or an object falling downward. Gravity decreases with distance according to the standard nonlinear second order differential equation

\[
\frac{d^2 r}{d^2} = -\frac{GM}{r^2}
\]
(def-quantitative-info
  (name initial-velocity-about-3000)
  (quantitative-initializations
    ;gravitational constant
    (G  (G*  (6.67e-11   6.67e-11)))
    ;Earth's mass
    (Earth-M (M*  (5.98e24   5.98e24 )))
    ;radius of Earth
    (r  (sea-level  (6.37e6   6.37e6 )))
    ;initial condition, less than escape velocity
    (dr/dt  (r*  (3000   3300 )))
  (envelopes ()))

Figure 9: Quantitative data describing known facts about the Earth, as well as the incompletely specified initial velocity \( \frac{dr}{dt} \) of a rocket (Figure 8) fired upward from the Earth's surface.

... each behavior. For the return behavior this was at time 153. For the escape behaviors, with \( T_1=\infty \) it occurred at time 1000 (Section 4.2.1). Constraint propagation on the result network for each behavior pruned the escape behaviors and improved the characterization of the return behavior somewhat. The pruning of an escape behavior is described in the return behavior was described in Subgoal 1 has been fully satisfied but the overall Go still knowlittle about how high the rock was, and narrow existing intervals, and in the return behavior. The

Subgoal 2

and

Subgoal 3

Step
<table>
<thead>
<tr>
<th>CONSTRAINT</th>
<th>INFERENCE</th>
<th>REASON</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Initial Conditions</strong></td>
<td>$R_{T0} = \text{SEA-LEVEL} = 6.37 \times 10^6$ m</td>
<td>Given</td>
</tr>
<tr>
<td>$\left( \frac{dR}{dt} \right)_T \in [3000, 3300]$ m/s</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Previously Inferred</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-k = -3.99 \times 10^4$</td>
<td>From $g$ and $M$ in Problem</td>
<td></td>
</tr>
<tr>
<td>$\left( \frac{d^2R}{dt^2} \right)_T = -9.83 \text{ m/s}^2$</td>
<td>From $R^2$ and $-k$ in Problem</td>
<td></td>
</tr>
<tr>
<td>$\text{Qualitative behavior and decreasing use that}$</td>
<td></td>
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</tr>
<tr>
<td>$\text{Solve equation (4) for } L \text{ in } q$, $q$, and $q$.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{Mean Value}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{Variable: } \frac{dR}{dt}$ (velocity, m/s upward)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{Variable: } R$ (radius, meters from the Earth’s center)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{TIME}$ $\tau_{T0} = 0$, $\tau_r = 1000$</td>
<td></td>
<td></td>
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<tr>
<td>$\text{Mean Value}$</td>
<td></td>
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<tr>
<td>$\text{Variable: } \frac{d^2R}{dt^2}$ (acceleration of gravity)</td>
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<tr>
<td>$\text{Variable: } \frac{dR}{dt}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{TIME}$ $\tau_{T0} = 0$, $\tau_r = 1000$</td>
<td></td>
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</tbody>
</table>

**Table 2:** (a) Excerpt of a trace showing how constraint on the rocket, after a state was interpolated at time: $T_1$. (b) A plot for a pruned escape behavior.
because behavior splitting creates new branches in the behavior tree and hence can lead to high computational complexity in subsequent simulation refinement. Thus, target interval splitting ("TTS") is used next, to try raising $T_2$ and lowering $T_1$.

Starting with the knowledge that $T_2 \in [326, \infty]$, Q3's implementation of target interval assumed $T_2 \in [326, 434]$, let the simulation settle via a constraining propagation and that settling led to an inconsistency (Table 2 exemplifies detecting $T_2 \notin [326, 434]$, so $[326, 434]$ was trimmed from the interval. Successfully ruling out $[326, 434]$, target interval $[326, 651]$. Subsequent to ruling out the adjacent interval $[482, 536]$, s
approaching point values in the limit if the model is specified with real valued initial conditions and model parameters. When the model is specified imprecisely with one or more intervals we are interested in stability, which intuitively means that if systems specifications are weak, widths of result intervals will be wider but only to a limited degree. We first deal followed by convergence, stability and finally termination for step sizes.

4.1 Carelessness

Numerical methods estimate answers, and interval methods bound the error so that the bounds safely contain the space of possible answers (epsilon may also include extraneous values, which may occur for

• Excess width. This is a well-known problem [Abelson & Sussman 1979]; see also e.g. Abelson & Sussman. The simplest such expression is \( X \) give a weaker answer. For example

\[ X \approx [1, 2] - [1, 2] = [0] \]

containing subtraction once. Hilbert's

either
For numerical simulation, convergence means improving point predictions all the way to full accuracy (Gear 1971). For interval simulations, convergence means narrowing interval enclosures all the way to correct point predictions (Bijgenraam 1981; Moore 1979 pp. 96–97; Lohner 1987 p. 261). Both senses apply in the limit as the step size of the simulation approaches zero. As the step size decreases, the total number of steps increases. The computation of simulations containing a large number of steps, together with round-off point arithmetic or the compensating extra width added in interval arithmetic, restricts convergence in practice. Nevertheless, simulation algorithms.

Our analysis builds on traditional analysis, such as Euler's method (Gear 1971; also see Appendix B) states:

Let \( Y' = \mathbf{F}(Y) \) be a system of \( m \) vector valued functions of \( Y \). We consider the component \( y_{i}(y) \) of vector \( y \) \( Y(y) \) \] such that

\[ f(y, y') \]

Let \( h \) be the maximum step size, and that each \( f \) in vector \( \mathbf{F} \) represent the amount of uncertainty such that

Given precise initial conditions, that \( \| Y_i \| \rightarrow 0 \) as \( h \rightarrow 0 \). This condition can be reduced arbitrarily close starting at time \( t \). The propagation of

be correct...
particular system suffers from excess width. For example, as Davis (1987) points out,

\[ x \in [1, 2] \implies \frac{x+1}{x} \in \left[ \frac{3}{2}, 2 \right] \]

but straightforward calculation (e.g., by hand or in Q3), gives

\[ \frac{x+1}{x} \in \left[ \frac{1, 2}{1, 2} +1 \right] = \left[ \frac{2, 3}{1, 2} \right] = \left[ \frac{2}{2}, 1 \right] = [1, 3]. \]

Thus this example demonstrates convergence despite excess width.

4.2.1 The infinitesimal step size assumption

Convergence as a theoretical property (both in numerical simulation and in the present case) assumes that the step size can be made infinitesimally small. We discuss this issue in detail below.

- If interpolation of each new time point can be done so as to reduce the size
  of the region in which convergence is desired, then continued
  strictly monotonic decrease in the maximum step size is

  Example: if the region of convergence is \([0, 1]\) such as
  and
  \[ 0.875 \] would not be allowed because a
  decrease in the interval before the gap \((0.75, 1)\) if a strictly

- The decrease in maximum step size
  strictly monotonic, but two smaller steps
  predefined continue

  Example: if
  \[ 0.1 \]
Figure 10: Example of convergence even though the interval calculations produce excess width. A simulation for \( \frac{dx}{dt} = \frac{x+1}{x} \) reveals a slightly concave down curve. Before step size refinement there are time points at 0 and \( \infty \). After refining the simulation by interpolating two new states at times \( t = 0.50 \) and \( t = 1.00 \), uncertainty in \( x \) increases rapidly, as shown by the two tall interval delimiters at \( t = 0.50 \) and \( t = 1.00 \). After refinement with ten interpolations, the time points significantly more constrained, as shown by the ten interval delimiters of intervals 0.10, 0.20, etc. Refinement with 100 interpolations leads to much shorter interval delimiters.
While convergence is universally recognized as an important theoretical property of simulation methods for continuous systems, it should be noted that pragmatically oriented uses of time point interpolation have not had convergence as a goal (Dvorak 1992; Kay 1996; this paper Section 5.3.2). Pragmatically oriented work shows that even one interpolation significantly improved quantitative bounds on model trajectories (Berleant detailed example).

43 Stability

In numerical simulation stability is, intuitively, the desirable characteristic starting values by a fixed amount produces a bounded change in the solution of a well-posed problem and sufficiently small step size.

Gear (1971 p. 56) defines stability more formally as

\[ \| y_n - \hat{y}_n \| \]

where \( y_0 \) and \( \hat{y}_0 \) are two sets of initial conditions and \( \| \) is a norm.

For a numerical simulation after \( n \) steps with \( y \) and \( \hat{y} \)

vector generalization of absolute equations containing some position

We adopt
4.3.1 Gap existence and creation

While step size refinement is stable, convergent, and correct, it can only run within a gap. The most common and important case is a gap starting at \( T_0 \in [0, 0] \). In particular:

- When the behavior has two qualitative time points \( T_0=0 \) and \( T_1=\infty \), the gap between \( T_0 \) and \( T_1 \) includes all positive finite values, allowing states to be interpolated at arbitrary times. Step size refinement is unconstrained.

- When \( 0 < T_1 < \infty \), the first gap is the open interval \((0, T_1)\), and step size refinement is unconstrained for time values in that interval. \( T_1 \) may also increase as the simulation is refined, increasing the size of the gap. (This occurred in Table 4.)

While often the requisite gaps will exist prior to step size refinement intervals in Phase I of the progressive simulation refinement may not, due to weak initial conditions. Q3 provides:

- Use target interval splitting. See Appendix.

- Use behavior splitting to force a particular behavior.

- Use another time step that resolves.

- Interpolate using a gap.

**Example:** Step size refinement is shown in Figure 11.

4.3.2 Termination

Constraint propagation implicitly have a finite number of other labels to terminate.
Figure 11: Hece we continuous simulation of an air conditioned dwelling. The time points in this simulation come from three sources: (1) qualitative simulation, which created time points T0, T1, T2, T3 and T4; (2) interpolations in the gap between T0 and T1, which created time point 1000; and (3) interpolations in gaps of model variable Inside Temperature, created time points K, K0, K1 and K2 at temperatures 79.5 and 81.5. Discontinuous plots are caused by transitions between models.
5.1 Impred predictions

By making quantitative inferences, semi-quantitative simulation can often prune qualitative behaviors that are plausible from a purely qualitative standpoint. A behavior is pruned when quantitative inference reveals that no interval is possible for some model variable at some time point (as seen in Table 2). Dalle Molle (1989) and Dalle Molle and Edgar (1991) used phase I of Q3 (Q2) purpose with two models of chemical engineering systems, the relatively simple but of two parallel first-order chemical processes, and the less simple adiabatic reactor.

Farquhar and Brajnik (1994) used phase I of Q3 in a system-level physics compiler”). They generated semi-quantitative models that were able to model and simulate a real hydroelectric plant levels for different water control scenarios.

5.2 Diagnosis

Semi-quantitative simulation can help diagnose models for which all behaviors are in one remaining fault model version (Dorak 1992).

5.3 Measurement

The concept of interpolation measurement partly power of this

5.3.1

Suppose the

\[ t = 3.75 \]


<table>
<thead>
<tr>
<th>Variable</th>
<th>Time point</th>
<th>Velocity</th>
<th>Height</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>400</td>
<td>3000</td>
<td>T1</td>
</tr>
<tr>
<td>No measurement</td>
<td>[6770, 18867]</td>
<td>[906, 17870]</td>
<td>[0, 0]</td>
</tr>
<tr>
<td>Wake measurement:</td>
<td>[6770, 18463]</td>
<td>[906, 14295]</td>
<td>[0, 0]</td>
</tr>
<tr>
<td>Height = 2000</td>
<td>[6770, 18293]</td>
<td>[906, 13844]</td>
<td>[0, 0]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variable</th>
<th>Time point</th>
<th>Acceleration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>400</td>
<td>3000</td>
</tr>
<tr>
<td>No measurement</td>
<td>[-1.59, -1.97]</td>
<td>[-1.37, -0.105]</td>
</tr>
<tr>
<td>Wake measurement:</td>
<td>[-1.59, -3.34]</td>
<td>[-1.37, -1.14]</td>
</tr>
<tr>
<td>Height = 2000</td>
<td>[-1.59, -3.67]</td>
<td>[-1.37, -1.2]</td>
</tr>
</tbody>
</table>

Table 3: Effects of different measurement strengths on predictions for Velocity, Height, and Acceleration of the rocket, at time points 400, 3000, and T1. The intervals for the “no measurement” condition are the same as in Figure 5. The effects of interpolating a state with a measurement condition are shown in the middle rows. A strong measurement introduces stronger quantitative information into the simulation.
The interpolation method of measurement interpretation contrasts with DeGoste's DATMI system (1991) and its precursor ATMI (Forbus 1986). A significant difference is that DATMI abstracts measurements into qualitative categories before using them whereas MIMIC and Q3 use the actual measured quantitative information. Hence DATMI loses quantitative information retained by MIMIC and Q3.

DATMI is intended for handling large numbers of measurements. The unmodified DATMI version is unwieldy for large numbers of measurements, but can be modified to circumscribe this limitation by propagating forward but not backward in time, and propagating forward by propagating forward but not backward in time, and propagating forward.

This was the approach taken by MIMIC.

5.4 Banding the probabilities of qualitative behaviors

Qualitative simulation alone can find all possible behaviors of a system but not their quantitative counterparts. Aiding quantitative information can help. Q3 was part of a system that included the qualitative behaviors of a fault tolerant system. Berleant functions (pdfs) were used instead of intervals to determine the qualitative behavior of a system. An interval representation of a pdf is informative than intervals. An interval represents zero beyond the interval endpoints, while the pdfs were first discrete and then continuous. Thus problem intervals and solution intervals are realized through the discretization of the pdfs.
NSIM (Kay and Kuijpers 1993) and SQSIM (Kay 1996) were developed in part to alleviate the wide bounds that Q3’s predecessor Q2 often infers. While NSIM sometimes provides better bounds than Q2 (Kay and Kuijpers 1993; Kuijpers 1994), sometimes NSIM’s results are poorer than Q2’s result which led to SQSIM which combines features of both NSIM and Q3. Kay (1996) describes SQSIM in detail but no comparison of its inferential ability to that of Q3 exists.

62 Numerical work

Forbus & Falkenhainer (1990, 1992) combined numerical and qualitative simulations in SIMGEN (SIMulator GENeration) system building on qualitative process theory. Forbus' work plays notable advantages.

1. Use of qualitatively inferred model transitions (e.g. weather and boiling commences) enabling automating simulations.

2. Causal ordering applied to qualitative model states.

Limitations of SIMGEN include (1) the need for precise numerical information, (2) output approximations, and (3) output approximations and often unreliable. While SIMGEN used numerical simulations as a base...
Figure 12: Fuzzy intervals. Sloping line segments indicate fuzzy regions. The lower the value of membership function \( \mu(z) \), the less the degree of membership for \( z \) in the fuzzy interval.

6.3.1 As with standard intervals, operations on fuzzy intervals can produce excess width

Figure 12 contains a very simple example of how the excess width problem in calculations on intervals has similar manifestations in calculations on fuzzy intervals. Values of \( x \) in the interval \([104, 106]\) are full members of fuzzy interval \( Z \) and those in the sloping angle are fuzzy members. Subtraction would give the region of full membership in the intersection \([104, 106] - [104, 106] = [-2, 2]\), the region of non-zero membership as \([102, 104]\), and fuzzy edges of constant slope. However, \( Z \) is perfectly correlated with \( x \) and has full membership at 0 and zero membership everywhere else.

Considering \( \mu(x) \) as an upper bound on the addresses the membership over-estimation results containing excess width.

Correlated fuzzy simulation cost of assuming all case of \( \mu(x) \) being 1.

Fuzzy
• *From interval simulation:* the guarantee that the trajectory of any real system conforming to an incompletely specified model is enclosed by one of the predicted semi-quantitative behaviors.

• *From interval simulation:* $h \to 0$ stability.

• *From interval simulation:* convergence as uncertainty in the quantitative specification of step size, both approach zero.

• *From qualitative and interval representations:* the ability to express and represent uncertainty, from partial knowledge.

The capabilities of $Q_3$ rely mostly on the following.

• *Step size refinement,* for adaptive reduction in step size by intermediate time points into a predicted behavior.

• *Propagation of interval labels* in constraint networks.

Examples of graphical output from $Q_3$ were produced involving the domains of prediction, displaying abilities of qualitative behavior. The significance of $Q_3$ to many quantitative methods.
Target Interval Splitting (TIS): Outline

**Given**
- \( Y = X - X \) and \( X \in [0, 1] \)

**Therefore:**
- \( Y \in [-1, 1] \) by constraint propagation (shown in this Figure).

**Objective:**
- Narrow \( Y \) (the target) further, by testing and ruling out pieces of its current interval as in Figure 14.

![Diagram](image)

Figure 13: A constraint network for the equation \( Y = X - X \) Given \( X \in [0, 1] \), constraint propagation concludes \( Y \in [-1, 1] \). This conclusion is correct, but excessively weak, and is strengthened in Figure 14.

An equation of the form \( a \circ b = c \) Constraints over more than three quantities in the mean value constraint, Section Transformation
Figure 14: **Target interval splitting narrows a target interval by ruling out pieces**.

The constraint network for \( Y = \bar{X} - X \) was shown in Figure 13. Target interval splitting tests the lower half of a target interval, \( Y \in [-1, 1] \) in this example, by setting the constraint network to 0. If the network settles successfully (i.e., has a solution), \([-1, -\frac{5}{2}]\) in this case, the lower eighth if necessary, etc. If the network has no solution, that sub-interval is tested. For the lowest quarter, \( Y \in [-1, -\frac{5}{2}] \), for the highest half, quarter is above. For \( Y = \bar{X} \) reaching, \([-\epsilon, \epsilon]\).
Proof: The proof has similarities with standard proofs of Euler's method (Gear 1971, 1976) and is also influenced by Moore (1979).
This equation (12) justify

\[ u(Y_n) \leq u(Y_{n-1}) + hL( Y_{n-1} + hF( M ) ). \]

7. Since F is Lipschitz and a natural interval extension, \( F( M ) \) is bounded.

Absolute value of an interval is the number...
**Theorem 1** Let $\dot{Y} = F(Y)$ be a system of first order differential equations, where $F$ is a vector of interval valued functions of $Y$. We consider some bounded subset $[l_0, h_i]$ of the reals such that for each component $Y_{(j)}$ of vector $Y$, $Y_{(j)}(t) \subseteq [l_0, h_i]$. We assume that $F(Y)$ is defined when each $Y_{(j)} \subseteq [l_0, h_i]$, and each $F_i$ in vector $F$ is the natural interval extension of a real rational function $f_i$.

Let $h$ be the maximum step size, let $\|Y_{\text{est}}\|$ represent the amount of uncertainty in the simulated estimate of $Y$ at interpolated time point $t = b$ as measured by its vector norm, and let $\|Y_0\|$ represent the amount of uncertainty in the initial conditions. Then there are constants $K_1$ and $\tilde{K}_h$ such that

$$\|Y_{\text{est}}\| \leq K_1 \|Y_0\| + \tilde{K}_h h.$$ 

1. **Higher order systems**: The proof of Lemma 1 extends to higher order systems. Each individual...
1. \( E_{\text{lower Heston}} \) and \( E_{\text{upper Heston}} \) differ only in the values of some constants. Then, \( G_{\text{lower Heston}} \equiv G_{\text{upper Heston}} \). Call this function \( G \).

Consider each constant \( c_i \) whose value differs between \( E_{\text{lower Heston}} \) and \( E_{\text{upper Heston}} \), the lower of the values and \( c_i \) the higher one. This instead of \( c_{\text{lower}} \) or \( c_{\text{upper}} \). By...


K Yap. *A System for Inteligently Guided Numerical Experimentation by Computer* (MIT Press, 1980) for systems. Course notes supplement 78.6, revised by Yang &

University of Texas at Austin (1978).