PROOF OF ALGORITHM 386 [A1]
GREATEST COMMON DIVISOR
OF n INTEGERS AND MULTIPLIERS*

by

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*[Gordon H. Bradley, Comm. ACM 13 (July 1970), 447]
ABSTRACT

Algorithm 386 is proved using the inductive assertion method. In the course of the proof, some errors were found and corrected. Some additional program changes are necessary for certain implementations of DØ statements.

KEY WORDS AND PHRASES: proof of algorithms, greatest common divisor, Euclidean algorithm, inductive assertion method.

CR CATEGORIES: 3.15, 4.42, 5.10, 5.24
INTRODUCTION

Subroutine GCDN, Algorithm 386 as described in [1,2], computes the greatest common divisor, IGCD, of n integers A(1),...,A(n) by using the Euclidean algorithm to compute first gcd(A(1),A(2)), then gcd(gcd(A(1),A(2)),A(3)), etc. It also computes integer multipliers Z(1),...,Z(n) such that IGCD = \sum_{i=1}^{n} A(i)Z(i). A proof that a modified version of GCDN performs these two tasks is given using the inductive assertion method.

PROOF PROCEDURE

The correctness of GCDN is proved by the inductive assertion method using a slight variation of one of the techniques described in [3]. Assertions concerning the progress of the computation are associated with various points in the program. The proof consists of showing that each assertion at a point is true each time control reaches that point in the program.

The inductive assertions are inserted as comments in the program below, assertion j.k being the kth assertion associated with statement j. A variable name with a zero subscript denotes the initial value of that variable (its value upon initiating the execution of GCDN) and a variable name without a subscript denotes the "current" value of that variable. The current value is the value of the variable just before execution of the program statement with which the inductive
assertion is associated. For example, \( N = N_0 \wedge A = A_0 \) is the fourth of four assertions associated with statement 2. This assertion states that the current value of \( N \) equals its initial value, and similarly, the current value of \( A \) equals its initial value.

The notation \( A = A_0 \), where \( A \) is an array name as in 2.4, is an abbreviation for \( A(i) = A_0(i) \) for all \( i \) within the dimension limits of \( A \). Also in writing the assertions, \( S(I,M,A,Z,ISIGN) \) is an abbreviation for the assertion

\[
gcd(A_0(1), \ldots , A_0(I)) = \sum_{k=M+1}^{I} \left( \prod_{j=k+1}^{I} A(j) \right) Z(k)A_0(k) + \left( \prod_{j=M+1}^{I} A(j) \right)(-2*ISIGN+1)A_0(M)
\]

and \( R(K,I,M,A,Z,ISIGN) \) is an abbreviation for

\[
IGCD = \sum_{j=M+1}^{I} Z(j)A_0(j) - \sum_{j=M+1}^{K-1} Z(j)A_0(j) + \sum_{k=M+1}^{K-1} \left( \prod_{j=k+1}^{K} A(j) \right) Z(k)A_0(k)
+ \left( \prod_{j=M+2}^{K} A(j) \right)(-2*ISIGN+1)A(M+1)A_0(M).
\]

Inductive assertions 1.1, 61.1 and 61.2 provide the formal statement of correctness for \( GCDN \). \( GCDN \) will be considered to be correct provided it has the following property: For every execution of \( GCDN \) initiated with 1.1 (\( 1 \leq N_0 \leq \dim(A) = \dim(Z) \)) true, and such that the execution terminates, then both 61.1 (\( IGCD = gcd[A_0(1), \ldots , A_0(N_0)] \)) and 61.2 (\( IGCD = \sum_{i=1}^{N_0} A_0(i)Z(i) \)) are true when the execution terminates. This is proved by the inductive assertion method, and hence, \( GCDN \) is correct. This property often is called "partial correctness". The term "correctness" then, is reserved for a program that not only is partially correct but also terminates for all executions satisfying the initial assertion. No formal proof is given here that \( GCDN \) always
terminates under initial assertion 1.1. However, this can be deduced from the bounds Bradley describes for the algorithm in [2]. In view of this, correctness does in fact simply amount to GCDN possessing the preceding property.

In a proof by inductive assertions, a verification condition is constructed for each control path. These verification conditions are mathematical conjectures that may be constructed in one of several forms [3,5]. The form used here is a variation of the path-forward form described in [3]. The first change is in the method of assigning alteration counters\(^1\) to the program variable names. Let \( n \) be the name of a program variable. Instead of using \( n_1 \) to denote the value of variable \( n \) at the beginning of a path, we simply use \( n \), and then \( n_1 \) denotes the value of \( n \) after the first time it appears on the left of an assignment, \( n_2 \) its value after the second time, ... This change in notation simply makes the verification condition more readable. The other change in the form of the verification condition is in the treatment of statements involving subscripted variables or division. In [3], it is suggested that an implicit test statement on the legality of subscripts be inserted before every statement containing a subscripted variable. Instead of this approach, we use a slightly different term in the verification condition. Consider, for example, the term due to statement 10, \( Z(M) = A(M) / IGCD \), along the path

\(^{1}\)An alteration counter is a subscript attached to each variable name that indicates how many times that variable has appeared on the left of an assignment statement along a particular control path.
(2,8,9,10,11,61). According to [3], the terms to be used in the verification condition for this statement are (under the new alteration counter convention)

\[ Z_1(M) = A(M) / \text{IGCD}_1 \]

\[ r \neq M \Rightarrow Z_1(r) = Z(r). \]

Instead of these terms, we shall use

\[(1 \leq M \leq \text{dim}(Z)) \land (1 \leq M \leq \text{dim}(A)) \land (\text{IGCD}_1 \neq 0) \Rightarrow Z_1(M) = A(M) / \text{IGCD}_1 \]

\[ r \neq M \Rightarrow Z_1(r) = Z(r). \]

By including the precondition in the first term, one is forced to prove that \( M \) is a legal subscript for both \( A \) and \( Z \) and that the division operation is defined before the term resulting from the assignment, \( Z_1(M) = A(M) / \text{IGCD}_1 \), can be used in further steps in the proof.

**MODIFICATIONS OF ORIGINAL ALGORITHM**

Some modifications to the original version of CCDN have been made solely to facilitate the proof of correctness. First, the original comments are removed and the inductive assertions are inserted. The original statement numbers have also been removed and each line of the program has been numbered.

To make explicit the interpretation of DO statements, all DO loops have been rewritten as IF loops. The statements which correspond
with the original D∅ loops have a single leading zero in the line number. D∅ statements have been assumed to consist of the following four steps. (1) Assign the control variable the value of the initial parameter. (2) Execute the body of the D∅ statement. (3) If control reaches the terminal statement, execute the terminal statement and increment the control variable by the incrementation parameter. (4) If the value of the control variable is less than or equal to the value of the terminal parameter, go back to 2, otherwise the D∅ is satisfied and execution continues out of the statement.

Also the RETURN statements have been replaced by G∅ T∅ statements that go to a single RETURN at statement 61.

CORRECTIONS TO THE ORIGINAL ALGORITHM

Three modifications of the program were necessitated by errors in the original algorithm. The statements in the code below which represent changes or corrections have their statement number field filled with leading zeros. Statements 9 and 10 are necessary in order to yield a positive greatest common divisor in the event that all elements of array A are zero except the last and it is negative. Statement 45 replaces the statement K = I-J+2 which is valid only if the first element of array A is non-zero. Statement 55 is necessary in the event that the greatest common divisor becomes one on the last element of array A. If $N_o < \text{dim}(Z)$, then statement 55 may be omitted,
however, this leads to the possibility of the value of the initial parameter of a DO statement being greater than the value of the terminal parameter.

For implementations in which DO statements are not handled as described above, some additional program modifications may be necessary. For example, according to the USA FORTRAN standard [4], at step 1 the value of the initial parameter must be less than or equal to the value of the terminal parameter and in step 4, if the DO is satisfied, the control variable becomes undefined. In subroutine GCDN, the only DO loop in which the value of the initial parameter may be greater than the value of the terminal parameter is the loop in statements 44 to 50. The program will give the correct results if this loop is executed once (as in the proof) or is bypassed, however if a fatal error will result, then the statement IF(MP2.GT.I)GO TO 51 should be inserted between statements 43 and 44. In many implementations, the control variable remains defined at the last value used in execution of the body of the DO when the DO is satisfied, in which case statement 42 may be omitted (as in the original version of the algorithm).

Statement 42 is necessary if the control variable becomes undefined, or if the control variable remains defined at its last value used in execution, incremented by the incrementation parameter (as in this proof).
PRESENTATION OF THE PROOF

The verification conditions for each control path are given in the tabular form described in [3]. First the path with which the verification condition is associated is given. The inductive assertion associated with the program statement at the beginning of the path is not rewritten with the verification condition since under the new alteration counter convention, the terms in the verification condition due to the assertion at the beginning of the path are identical with the assertion itself. The terms above the line in the verification condition are the terms constructed from the program statements along the control path. These terms are numbered with their respective program statement numbers. The terms below the line are the ones constructed from the inductive assertions associated with the program statement at the end of the path.

According to the inductive assertion method, if for every verification condition it can be shown that each term below the line follows from the terms above the line, including the inductive assertion at the beginning of the path and assertion 1.1, then the program is correct. For the sake of brevity, we have exhibited only those proofs which require more than a few straightforward steps. It is assumed throughout these proofs that all arithmetic operations are integer operations of arbitrarily high precision. Also we use the following properties of gcd:
P1. \( \gcd(0,0) = 0 \)

P2. \( \gcd(1,n) = 1 \)

P3. \( \gcd(0,n) = |n| \)

P4. \( \gcd(a_1, \ldots, a_n) = \gcd(\gcd(a_1, \ldots, a_{n-1}), a_n) \)

P5. If \( am + bn = c \), then \( \gcd(a,b) = \gcd(a,c) \)

P6. \( \gcd(m,n) = \gcd(|m|,n) \)

P7. If \( \gcd(m,n) = 0 \), then \( m = n = 0 \).

P8. \( \gcd(m,n) = \gcd(n,m) \)

P9. \( \gcd(n) = |n| \)
REFERENCES


SUBROUTINE GCDN(N,A,Z,IGCD)
DIMENSION A(dim(A)),Z(dim(Z))
INTEGER A,Z,C1,C2,Y1,Y2,Q
C 1.1 1 ≤ N_o ≤ dim(A) = dim(Z)
 01 M = 1
C 2.1 1 ≤ M ≤ N_o
C 2.2 1 ≤ i ≤ M - 1 ⇒ A(i) = 0
C 2.3 1 ≤ i ≤ M - 1 ⇒ Z(i) = 0
C 2.4 N = N_o ∧ A = A_o
  2 IF(A(M).NE.0) G0 T0 8
  3 Z(M) = 0
  04 M = M + 1
  05 IF(M.LE.N) G0 T0 2
  6 IGCD = 0
  7 G0 T0 61
  8 IF(M.NE.N) G0 T0 12
  00009 IGCD = IABS(A(M))
  00010 Z(M) = A(M) / IGCD
  11 G0 T0 61
  12 MP1 = M + 1
  13 MP2 = M + 2
  14 ISIGN = 0
  15 IF(A(M).GE.0) G0 T0 18
  16 ISIGN = 1
  17 A(M) = -A(M)
  18 C1 = A(M)
  019 I = MP1
C 20.1 1 ≤ i ≤ M - 1 ⇒ Z(i) = 0
C 20.2 N = N_o ∧ MP1 = M + 1 ∧ MP2 = M + 2 ∧ 0 ≤ ISIGN ≤ 1
C 20.3 2 ≤ M + 1 ≤ I ≤ N_o
C 20.4 C1 = A(M) = gcd(A_o(1),...,A_o(I-1)) ≠ 0
C 20.5 k ≤ I ⇒ A(k) = A_o(k)
C 20.6 S(I-1,M,A,Z,ISIGN)
20 IF(A(I).NE.0) G0 T0 23
21 A(I) = 1
22 Z(I) = 0
   G0 T0 39
23 Y1 = 1
24 Y2 = 0
25 C2 = IABS(A(I))
26 Q = C2 / C1
27 C2 = C2 - Q * C1
C 28.1 1 <= i <= M - 1 D Z(i) = 0
C 28.2 N = N_o A MP1 = M + 1 A MP2 = M + 2 A 0 <= ISIGN <= 1
C 28.3 2 <= M + 1 <= I <= N_o
C 28.4 A(M) = gcd(A_o(1),...,A_o(I-1)) != 0
C 28.5 k <= I D A(k) = A_o(k)
C 28.6 gcd(C1,C2) = gcd(A_o(I),...,A_o(I))
C 28.7 A_o(I) != 0
C 28.8 S(I-1,M,A,Z,ISIGN)
C 28.9 A(I) divides C1 - Y1 * A(M)
C 28.10 A(I) divides C2 - (Y2 - Q * Y1) * A(M)
28 IF(C2.EQ.0) G0 T0 36
29 Y2 = Y2 - Q * Y1
30 Q = C1 / C2
31 C1 = C1 - Q * C2
32 IF(C1.EQ.0) G0 T0 34
33 Y1 = Y1 - Q * Y2
   G0 T0 26
34 C1 = C2
35 Y1 = Y2
36 Z(I) = (C1 - Y1 * A(M)) / A(I)
37 A(I) = Y1
38 A(M) = C1
C 39.1  \[ 1 \leq i \leq M - 1 \implies Z(i) = 0 \]
C 39.2  \[ N = N_0 \land MP1 = M + 1 \land MP2 = M + 2 \land 0 \leq ISIGN \leq 1 \]
C 39.3  \[ 2 \leq M + 1 \leq N \]
C 39.4  \[ C_1 = A(M) = \gcd(A_o(1), \ldots, A_o(I)) \neq 0 \]
C 39.5  \[ k > I \implies A(k) = A_o(k) \]
C 39.6  \[ S(I, M, A, Z, ISIGN) \]

39  IF(C_1.EQ.1) GØ TØ 55

40  I = I + 1

040  IF(I.LE.N) GØ TØ 20

0042  I = N

43  IGCD = A(M)

044  J = MP2

0045  K = I - J + MP1

46  KK = K + 1

47  Z(K) = Z(K) * A(KK)

48  A(K) = A(K) * A(KK)

C 49.1  \[ 1 \leq i \leq M - 1 \implies Z(i) = 0 \]
C 49.2  \[ I + 1 \leq i \leq N \implies Z(i) = 0 \]
C 49.3  \[ MP1 = M + 1 \land MP2 = M + 2 \land 0 \leq ISIGN \leq 1 \]
C 49.4  \[ 2 \leq M + 1 \leq I \leq N \land MP2 \leq J \]
C 49.5  \[ K = I - J + MP1 \]
C 49.6  \[ R(K, I, M, A, Z, ISIGN) \]
C 49.7  \[ IGCD = \gcd(A_o(1), \ldots, A_o(N_o)) \]

049  J = J + 1

050  IF(J.LE.I) GØ TØ 45

51  Z(M) = A(MP1)

52  IF(ISIGN.EQ.0) GØ TØ 54

53  Z(M) = -Z(M)

54  GØ TØ 61

00055  IF(I.EQ.N) GØ TØ 43

56  IP1 = I + 1

057  J = IP1

58  Z(J) = 0
C 59.1  1 ≤ i ≤ M - 1  ⇒  Z(i) = 0
C 59.2  I + 1 ≤ i ≤ J  ⇒  Z(i) = 0
C 59.3  N = N_o  ∧  MP_l = M + 1  ∧  MP_2 = M + 2  ∧  0 ≤ I ≤ J ≤ N_o
C 59.4  2 ≤ M + 1 ≤ I ≤ J ≤ N_o
C 59.5  A(M) = gcd(A_o(1), ..., A_o(I)) = gcd(A_o(1), ..., A_o(N_o))
C 59.6  S(I,M,A,Z,ISIGN)

059  J = J + 1
060  IF(J.LE.N) GOTO 58
     GOTO 43
C 61.1  IGCD = gcd(A_o(1), ..., A_o(N_o))
C 61.2  IGCD = \sum_{i=1}^{N_o} A_o(i)Z(i)
61  RETURN
END
Path (1,2).

1 \hspace{1cm} M_1 = 1

\hspace{1cm}

2.1 \hspace{1cm} 1 \leq M_1 \leq N_o
2.2 \hspace{1cm} 1 \leq i \leq M_1 - 1 \Rightarrow A_o(i) = 0
2.3 \hspace{1cm} 1 \leq i \leq M_1 - 1 \Rightarrow Z_o(i) = 0
2.4 \hspace{1cm} N_o = N_o \& A_o = A_o

Path (2,3,4,5,2).

2 \hspace{1cm} 1 \leq M \leq \text{dim}(A) \Rightarrow A(M) = 0
3a \hspace{1cm} 1 \leq M \leq \text{dim}(Z) \Rightarrow Z_1(M) = 0
\ \ \ b \hspace{1cm} r \neq M \Rightarrow Z_1(r) = Z(r)
4 \hspace{1cm} M_1 = M + 1
5 \hspace{1cm} M_1 \leq N

\hspace{1cm}

2.1' \hspace{1cm} 1 \leq M_1 \leq N_o
2.2' \hspace{1cm} 1 \leq i \leq M_1 - 1 \Rightarrow A(i) = 0
2.3' \hspace{1cm} 1 \leq i \leq M_1 - 1 \Rightarrow Z_1(i) = 0
2.4' \hspace{1cm} N = N_o \& A = A_o
Path (2, 3, 4, 5, 6, 61).

2 \[ 1 \leq M \leq \dim(A) \supseteq A(M) = 0 \]
3a \[ 1 \leq M \leq \dim(Z) \supseteq Z_1(M) = 0 \]
   \[ b \quad r \neq M \supseteq Z_1(r) = Z(r) \]
4 \[ M_1 = M + 1 \]
5 \[ M_1 > N \]
6 \[ IGCD_1 = 0 \]

61.1 \[ IGCD_1 = \gcd(A_0(1), \ldots, A_0(N_0)) \]
    Proof. Note that all elements of A are zero.

61.2 \[ IGCD_1 = \sum_{i=1}^{N_0} A_0(i)Z_1(i) \]

Path (2, 8, 9, 10, 61).

2 \[ 1 \leq M \leq \dim(A) \supseteq A(M) \neq 0 \]
8 \[ M = N \]
9 \[ 1 \leq M \leq \dim(A) \supseteq IGCD_1 = IABS(A(M)) \]
10a \[ (1 \leq M \leq \dim(Z)) \land (1 \leq M \leq \dim(A)) \land (IGCD_1 \neq 0) \supseteq Z_1(M) = A(M) / IGCD_1 \]
   \[ b \quad r \neq M \supseteq Z_1(r) = Z(r) \]

61.1 \[ IGCD_1 = \gcd(A_0(1), \ldots, A_0(N_0)) \]
    Proof. \( A_0(N_0) \) is the only non-zero element.

61.2 \[ IGCD_1 = \sum_{i=1}^{N_0} A_0(i)Z_1(i) \]
Path \((2,8,12,13,14,15,16,17,18,19,20)\).

\[
\begin{align*}
2 & \quad 1 \leq M \leq \dim(A) \implies A(M) \neq 0 \\
8 & \quad M \neq N \\
12 & \quad MP_1 = M + 1 \\
13 & \quad MP_2 = M + 2 \\
14 & \quad ISIGN_1 = 0 \\
15 & \quad 1 \leq M \leq \dim(A) \implies A(M) \leq 0 \\
16 & \quad ISIGN_2 = 1 \\
17a & \quad 1 \leq M \leq \dim(A) \implies A_1(M) = -A(M) \\
& \quad b \quad r \neq M \implies A_1(r) = A(r) \\
18 & \quad 1 \leq M \leq \dim(A) \implies Cl_1 = A_1(M) \\
19 & \quad I_1 = MP_1
\end{align*}
\]

\[
\begin{align*}
20.1 & \quad 1 \leq i \leq M-1 \implies Z(i) = 0 \\
20.2 & \quad N = N_o \land MP_1 = M + 1 \land MP_2 = M + 2 \land 0 \leq ISIGN_2 \leq 1 \\
20.3 & \quad 2 \leq M + 1 \leq I_1 \leq N_o \\
20.4 & \quad Cl_1 = A_1(M) = \gcd(A_o(1),...,A_o(I_1-1)) \neq 0 \\
\text{Proof.} & \quad Cl_1 = A_1(M) \neq 0 \text{ from 2,18.} \\
& \quad \gcd(A_o(1),...,A_o(I_1-2)) = 0 \text{ from 2.2,12,19.} \\
& \quad \gcd(A_o(1),...,A_o(I_1-1)) = \left| A_o(I_1-1) \right| = \left| A_o(M) \right|. \\
& \quad \gcd(A_o(1),...,A_o(I_1-1)) = A_1(M) \text{ from 15,17.} \\
20.5 & \quad k \geq I_1 \implies A_1(k) = A_o(k) \\
20.6 & \quad S(I_1-1,M,A_1,Z,ISIGN_2) \\
\text{Proof.} & \quad \gcd(A_o(1),...,A_o(I_1-1)) = -A_o(M) \text{ from 20.4} \\
& \quad I_1-1 \leq M + 1 \\
& \quad \gcd(A_o(1),...,A_o(I_1-1)) = \sum_{k=M+1}^{I_1-1} \left( \frac{I_1-1}{j} \right) \frac{A_1(j)}{A_o(k)} \\
& \quad \quad + \left( \frac{I_1-1}{j} \right) A_1(j) (-2*ISIGN_2 + 1)A_o(M)
\end{align*}
\]
Path \((2, 8, 12, 13, 14, 15, 18, 19, 20)\).

\begin{align*}
2 & \quad 1 \leq M \leq \dim(A) \implies A(M) \neq 0 \\
8 & \quad M \neq N \\
12 & \quad \text{MP}_1 = M + 1 \\
13 & \quad \text{MP}_2 = M + 2 \\
14 & \quad \text{SIGN}_1 = 0 \\
15 & \quad 1 \leq M \leq \dim(A) \implies A(M) \geq 0 \\
18 & \quad 1 \leq M \leq \dim(A) \implies C_1 = A(M) \\
19 & \quad I_1 = \text{MP}_1 \\
\end{align*}

\begin{align*}
20.1 & \quad 1 \leq i \leq M - 1 \implies z(i) = 0 \\
20.2 & \quad N = N_o \land \text{MP}_1 = M + 1 \land \text{MP}_2 = M + 2 \land 0 \leq \text{SIGN}_1 \leq 1 \\
20.3 & \quad 2 \leq M + 1 \leq I_1 \leq N_o \\
20.4 & \quad C_1 = A(M) = \gcd(A_o(1), \ldots, A_o(I_1 - 1)) \neq 0 \\
\text{Proof.} & \quad \text{See 20.4 on previous path.} \\
20.5 & \quad k \leq I_1 \implies A(k) = A_o(k) \\
20.6 & \quad S(I_1 - 1, M, A, Z, \text{SIGN}_1) \\
\text{Proof.} & \quad \text{See 20.6 on previous path.}
\end{align*}
Path (20, 21, 22, 39).

20 \quad 1 \leq I \leq \dim(A) \Rightarrow A(I) = 0

21a \quad 1 \leq I \leq \dim(A) \Rightarrow A_1(I) = 1

\quad b \quad r \neq I \Rightarrow A_1(r) = A(r)

22a \quad 1 \leq I \leq \dim(Z) \Rightarrow Z_1(I) = 0

\quad b \quad r \neq I \Rightarrow Z_1(r) = Z(r)

39.1 \quad 1 \leq i \leq M-1 \Rightarrow Z_1(i) = 0

39.2 \quad N = N_o \land MP1 = M + 1 \land MP2 = M + 2 \land 0 \leq \text{ISIGN} \leq 1

39.3 \quad 2 \leq M + 1 \leq I \leq N_o

39.4 \quad C_1 = A_1(M) = \gcd(A_o(1), \ldots, A_o(I)) \neq 0

Proof. \quad C_1 = \gcd(A_o(1), \ldots, A_o(I)) \neq 0 \text{ from 20.4, 20.}

\quad C_1 = A_1(M) = \gcd(A_o(1), \ldots, A_o(I)) \neq 0 \text{ from 20.4 since } A_1(M) = A(M) \text{ from 21b.}

39.5 \quad k > I \Rightarrow A_1(k) = A_o(k)

Proof. \quad \text{See 20.5, 21b.}

39.6 \quad S(I, M, A_1, Z_1, \text{ISIGN})

Proof. \quad S(I-1, M, A_1, Z_1, \text{ISIGN}) \text{ from 20.6, 21b, 22b.}

\quad A_o(I) = 0 \text{ from 20.5, 20.}

\quad \prod_{j=k+1}^{I-1} A_1(j) = \prod_{j=k+1}^{I} A_1(j) \text{ from 21a}

\quad \gcd(A_o(1), \ldots, A_o(I)) = \sum_{k=M+1}^{I-1} \left( \prod_{j=k+1}^{I} A_1(j) \right) Z_1(k) A_o(k)

\quad + \left( \prod_{j=M+1}^{I} A_1(j) \right) (-2 \times \text{ISIGN} + 1) A_o(M) \text{ from above statements}

\quad S(I, M, A_1, Z_1, \text{ISIGN}) \text{ since } Z_1(I) = 0 \text{ from 22a.}
Path \((20,23,24,25,26,27,28)\).

\[ 1 \leq I \leq \dim(A) \Rightarrow A(I) \neq 0 \]

\[ Y_{1_1} = 1 \]

\[ Y_{2_1} = 0 \]

\[ 1 \leq I \leq \dim(A) \Rightarrow C_{2_1} = \text{IABS}(A(I)) \]

\[ C_1 \neq 0 \Rightarrow Q_1 = C_{2_1} / C_1 \]

\[ C_{2_2} = C_{2_1} - Q_1 \times C_1 \]

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28.1 \[ 1 \leq i \leq M-1 \Rightarrow Z(i) = 0 \]

28.2 \[ N = N_0 \land MP_1 = M + 1 \land MP_2 = M + 2 \land 0 \leq \text{ISIGN} \leq 1 \]

28.3 \[ 2 \leq M + 1 \leq I \leq N_0 \]

28.4 \[ A(M) = \gcd(A_{o_1}(1), \ldots, A_{o_1}(I-1)) \neq 0 \]

28.5 \[ k \leq I \Rightarrow A(k) = A_{o_1}(k) \]

28.6 \[ \gcd(C_1, C_{2_2}) = \gcd(A_{o_1}(1), \ldots, A_{o_1}(I)) \]

Proof. \[ \gcd(C_1, C_{2_2}) = \gcd(C_1, C_{2_1}) \text{ from } 27 \text{ and } P5 \]

\[ = \gcd(C_1, A_{o_1}(I)) \text{ from } 25 \]

\[ = \gcd(A_{o_1}(1), \ldots, A_{o_1}(I)) \text{ from } 20.4, P4 \]

28.7 \[ A_{o_1}(I) \neq 0 \]


28.8 \[ S(I-1,M,A,Z,\text{ISIGN}) \]

28.9 \[ A(I) \text{ divides } C_1 - Y_{1_1} \times A(M) \]

Proof. \[ C_1 = A(M) \text{ from } 20.4 \]

\[ Y_{1_1} = 1 \text{ from } 23 \]

\[ A(I) \text{ divides } 0 = C_1 - Y_{1_1} \times A(M) \]

28.10 \[ A(I) \text{ divides } C_{2_2} - (Y_{2_1} - Q_1 \times Y_{1_1}) \times A(M) \]

Proof. \[ C_{2_2} - (Y_{2_1} - Q_1 \times Y_{1_1}) \times A(M) = C_{2_1} - Q_1 \times C_1 + Q_1 \times A(M) \]

\[ = C_{2_1} = |A(I)| \text{ since } C_1 = A(M) \]

\[ \text{from } 23,24,27. \]

\[ = C_{2_1} = |A(I)| \text{ from } 20.4,25. \]

\[ A(I) \text{ divides } |A(I)| = C_{2_2} - (Y_{2_1} - Q_1 \times Y_{1_1}) \times A(M) \]
Path $(28, 36, 37, 38, 39)$.

28  \[ C_2 = 0 \]

36a  \[ 1 \leq I \leq \dim(Z) \leq M \leq \dim(A) \leq 1 \leq I \leq \dim(A) \land A(I) \neq 0 \Rightarrow Z_1(I) = (C_1 - Y_1 \ast A(M)) / A(I) \]

b  \[ r \neq I \Rightarrow Z_1(r) = Z(r) \]

37a  \[ 1 \leq I \leq \dim(A) \Rightarrow A_1(I) = Y_1 \]

b  \[ r \neq I \Rightarrow A_1(r) = A(r) \]

38a  \[ 1 \leq M \leq \dim(A) \Rightarrow A_2(M) = C_1 \]

b  \[ r \neq M \Rightarrow A_2(r) = A_1(r) \]

39.1  \[ 1 \leq i \leq M - 1 \Rightarrow Z_1(i) = 0 \]

39.2  \[ N = N_0 = M + 1 \land M_2 = M + 2 \neq 0 \Rightarrow \text{ISIGN} \neq 1 \]

39.3  \[ 2 \leq M + 1 \leq \text{N}_0 \]

39.4  \[ C_1 = A_2(M) = \gcd(A_0(1), \ldots, A_0(I)) \neq 0 \]

Proof. \[ \gcd(A_0(1), \ldots, A_0(I-1)) \neq 0 \text{ from 28.4} \]

\[ \gcd(A_0(1), \ldots, A_0(I)) \neq 0 \text{ from P7} \]

\[ C_1 = A_2(M) = \gcd(A_0(1), \ldots, A_0(I)) \neq 0 \text{ from 28.6, 28, 38a, P3} \]

39.5  \[ k \neq I \Rightarrow A_2(k) = A_0(k) \]

Proof. See 28.5, 37b, 38b.

39.6  \[ S(I, M, A_2, Z_1, \text{ISIGN}) \]

Proof. (a) \[ S(I-1, M, A, Z, \text{ISIGN}) \text{ from 28.8} \]

(b) \[ A(M) = \sum_{k=M+1}^{I-1} \left( \prod_{j=k+1}^{I-1} A_2(j) \right) Z_1(k) A_0(k) \]

\[ + \left( \prod_{j=M+1}^{I-1} A_2(j) \right) (-2 \ast \text{ISIGN} + 1) A_0(M) \text{ from 28.4, 36b, 37b, 38b.} \]

(c) \[ C_1 = Z_1(I) \ast A(I) + Y_1 \ast A(M) \text{ from 28.9, 36} \]

(d) \[ C_1 = Z_1(I) \ast A_0(I) + A_2(I) \ast A(M) \text{ from 28.5, 37a, 38b} \]

(e) \[ \gcd(A_0(1), \ldots, A_0(I)) = Z_1(I) \ast A_0(I) + A_2(I) \ast A(M) \text{ from 28.6, 28, P3.} \]

(f) \[ S(I, M, A_2, Z_1, \text{ISIGN}) \text{ substitute (b) into (e).} \]
Path \((28,29,30,31,32,33,26,27,28)\).

28\hspace{1cm} C_2 \neq 0

29\hspace{1cm} Y_{2_1} = Y_2 - Q \ast Y_1

30\hspace{1cm} C_2 \neq 0 \implies Q_1 = C_1 / C_2

31\hspace{1cm} C_1 = C_1 - Q_1 \ast C_2

32\hspace{1cm} C_1 \neq 0

33\hspace{1cm} Y_{1_1} = Y_1 - Q_1 \ast Y_{2_1}

26\hspace{1cm} C_1 \neq 0 \implies Q_2 = C_2 / C_1

27\hspace{1cm} C_2 = C_2 - Q_2 \ast C_1

\underline{28.1'} \hspace{1cm} 1 \leq i \leq M-1 \implies Z(i) = 0

\underline{28.2'} \hspace{1cm} N = N_o \wedge MP1 = M + 1 \wedge MP2 = M + 2 \wedge 0 \leq ISIGN \leq 1

\underline{28.3'} \hspace{1cm} 2 \leq M + 1 \ which \implies ISIGN

\underline{28.4'} \hspace{1cm} A(M) = \gcd(A_o(1),\ldots,A_o(I-1)) \neq 0

\underline{28.5'} \hspace{1cm} k \geq I \implies A(k) = A_o(k)

\underline{28.6'} \hspace{1cm} \gcd(C_1,C_2) = \gcd(A_o(1),\ldots,A_o(I))

\begin{proof}
See 28.6, 27, 31.
\end{proof}

\underline{28.7'} \hspace{1cm} A_o(I) \neq 0

\underline{28.8'} \hspace{1cm} S(I-1,M,A,Z,ISIGN)

\underline{28.9'} \hspace{1cm} A(I) \ divides \ C_1 - Y_{1_1} \ast A(M)

\begin{proof}
C_1 - Y_{1_1} \ast A(M) = (C_1 - Q_1 \ast C_2 - (Y_1 - Q_1 \ast Y_{2_1}) \ast A(M) \ from \ 31, 33

= (C_1 - Y_1 \ast A(M)) - Q_1(C_2 - (Y_2 - Q_1 \ast Y_1) \ast A(M)) \ from \ 29

A(I) \ divides \ (C_1 - Y_1 \ast A(M)) - Q_1(C_2 - (Y_2 - Q_1 \ast Y_1) \ast A(M)) \ from \ 28.9, 28.10.
\end{proof}

\underline{28.10'} \hspace{1cm} A(I) \ divides \ C_2 - (Y_{2_1} - Q_2 \ast Y_{1_1}) \ast A(M)

\begin{proof}
C_2 - (Y_{2_1} - Q_2 \ast Y_{1_1}) \ast A(M) = (C_2 - Q_2 \ast C_1 - (Y_2 - Q_1 \ast Y_1) - Q_2 \ast Y_{1_1} \ast A(M)

from \ 27, 29.

= (C_2 - (Y_2 - Q_1 \ast Y_1) \ast A(M)) - Q_2(C_1 - Y_{1_1} \ast A(M))

A(I) \ divides \ (C_2 - (Y_2 - Q_1 \ast Y_1) \ast A(M)) - Q_2(C_1 - Y_{1_1} \ast A(M))

from \ 28.10, 28.9'.

Path (28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 39).

28  \[ C_2 \neq 0 \]
29  \[ Y_{21} = Y_{2-Q} \ast Y_1 \]
30  \[ C_2 \neq 0 \Rightarrow Q_1 = C_1 \div C_2 \]
31  \[ Q_1 = C_1 - Q_1 \ast C_2 \]
32  \[ Q_1 = 0 \]
33  \[ C_{21} = C_2 \]
35  \[ Y_{11} = Y_{21} \]
36a  \[ 1 \leq I \leq \dim(Z) \land 1 \leq M \leq \dim(A) \Rightarrow 1 \leq I \leq \dim(A) \land A(I) \neq 0 \Rightarrow Z_1(I) = (C_{12} - Y_{11} \ast A(M)) \div A(I) \]
37a  \[ r \neq I \Rightarrow Z_1(r) = Z(r) \]
38a  \[ 1 \leq M \leq \dim(A) \Rightarrow A_1(M) = Y_{11} \]
39.1  \[ 1 \leq i \leq M \Rightarrow Z_1(i) = 0 \]

Proof. See 36b.

39.2  \[ N = \text{N}_0 \land MP1 = M + 1 \land MP2 = M + 2 \land 0 \leq \text{ISIGN} \leq 1 \]
39.3  \[ 2 \leq M + 1 \leq I \leq \text{N}_0 \]
39.4  \[ C_{21} = A_2(M) = \gcd(A_o(1), \ldots, A_o(I)) \neq 0 \]

Proof. \[ \gcd(C_1, C_2) = \gcd(A_o(1), \ldots, A_o(I)) \] from 28.6
\[ \gcd(C_{11}, C_2) = \gcd(A_o(1), \ldots, A_o(I)) \] from 31, P5
\[ C_2 = \gcd(A_o(1), \ldots, A_o(I)) \neq 0 \] from 28, 32
\[ C_{21} = A_2(M) = \gcd(A_o(1), \ldots, A_o(I)) \neq 0 \] from 34, 38a

39.5  \[ k \Rightarrow I \Rightarrow A_2(k) = A_o(k) \]

Proof. See 28.5, 37b, 38b.
39.6  \( S(I,M,A_2,Z_1,\text{ISIGN}) \)

Proof. (a) \( S(I-1,M,A,Z,\text{ISIGN}) \) from 28.8

\[
A(M) = \sum_{k=M+1}^{I-1} \left( \prod_{j=k+1}^{I-1} A_2(j) \right) Z_1(k) A_0(k)
\]

\[
+ \left( \prod_{j=M+1}^{I-1} A_2(j) \right) (-2*\text{ISIGN} + 1) A_0(M) \text{ from 28.4, 36b,37b,38b.}
\]

(c) \( A(I) \) divides \( C2-(Y2-Q \ast Y1) \ast A(M) \) from 28.10

(d) \( A(I) \) divides \( C1_2-Y1_1 \ast A(M) \) from 29,34,35

(e) \( C1_2 = Z_1(I) \ast A(I) + Y1_1 \ast A(M) \) from 36a

(f) \( C2 = Z_1(I) \ast A(I) + A_2(I) \ast A(M) \) from 34,37a,38b

(g) \( \gcd(C1,C2) = Z_1(I) \ast A(I) + A_2(I) \ast A(M) \) from 31,32,P3,P5

(h) \( \gcd(A_0(I), \ldots, A_0(I)) = Z_1(I) \ast A(I) + A_2(I) \ast A(M) \) from 28.6

(i) \( S(I,M,A_2,Z_1,\text{ISIGN}) \) substitute (b) into (h).
Path \((39,40,41,20)\).

\[
\begin{align*}
39 & \quad C_1 \neq 1 \\
40 & \quad I_1 = I + 1 \\
41 & \quad I_1 \leq N
\end{align*}
\]

\[
\begin{align*}
20.1 & \quad 1 \leq i \leq M-1 \Rightarrow Z(i) = 0 \\
20.2 & \quad N = N_0 \wedge MP1 = M + 1 \wedge MP2 = M + 2 \wedge 0 \leq ISIGN \leq 1 \\
20.3 & \quad 2 \leq M + 1 \leq I_1 \leq N_0 \\
20.4 & \quad C_1 = A(M) = \gcd(A_0(1), \ldots, A_0(I_1-1)) \neq 0 \\
20.5 & \quad k \geq I_1 \Rightarrow A(k) = A_0(k) \\
20.6 & \quad S(I_1-1, M, A, Z, ISIGN)
\end{align*}
\]

Path \((39,55,56,57,58,59)\).

\[
\begin{align*}
39 & \quad C_1 = 1 \\
55 & \quad I \neq N \\
56 & \quad IP1_1 = I + 1 \\
57 & \quad J_1 = IP1_1 \\
58a & \quad 1 \leq J_1 \leq \dim(Z) \Rightarrow Z_1(J_1) = 0 \\
b & \quad r \neq J_1 \Rightarrow Z_1(r) = Z(r)
\end{align*}
\]

\[
\begin{align*}
59.1 & \quad 1 \leq i \leq M-1 \Rightarrow Z_1(i) = 0 \\
59.2 & \quad I + 1 \leq i \leq J_1 \Rightarrow Z_1(i) = 0 \\
\text{Proof.} & \quad \text{See 56,57,58.}
\end{align*}
\]

\[
\begin{align*}
59.3 & \quad N = N_0 \wedge MP1 = M + 1 \wedge MP2 = M + 2 \wedge 0 \leq ISIGN \leq 1 \\
59.4 & \quad 2 \leq M + 1 \leq I \leq J_1 \leq N_0 \\
59.5 & \quad A(M) = \gcd(A_0(1), \ldots, A_0(I)) = \gcd(A_0(1), \ldots, A_0(N_0)) \\
\text{Proof.} & \quad \text{See 39,4,39,P2}
\end{align*}
\]

\[
\begin{align*}
59.6 & \quad S(I, M, A, Z_1, ISIGN) \\
\text{Proof.} & \quad \text{See 58b.}
\end{align*}
\]
Path \((39,55,43,44,45,46,47,48,49)\).

\[39\]
\[C_1 = 1\]

\[55\]
\[I = N\]

\[43\]
\[1 \leq M \leq \dim(A) \Rightarrow \text{IGCD}_1 = A(M)\]

\[44\]
\[J_1 = \text{MP2}\]

\[45\]
\[K_1 = I - J_1 + \text{MP1}\]

\[46\]
\[KK_1 = K_1 + 1\]

\[47a\]
\[1 \leq K_1 \leq \dim(Z) \Rightarrow A_1(K_1) = A(K_1) \ast A(KK_1)\]

\[b\]
\[r \neq K_1 \Rightarrow Z_1(r) = Z(r)\]

\[48a\]
\[1 \leq K_1 \leq \dim(A) \Rightarrow A_1(K_1) = A(K_1) \ast A(KK_1)\]

\[b\]
\[r \neq K_1 \Rightarrow A_1(r) = A(r)\]

\[49.1\]
\[1 \leq i \leq M-1 \Rightarrow Z_1(i) = 0\]

Proof. See 47b.

\[49.2\]
\[I + 1 \leq i \leq N \Rightarrow Z_1(i) = 0\]

Proof. See 55.

\[49.3\]
\[\text{MP1} = M + 1 \land \text{MP2} = M + 2 \land 0 \leq \text{ISIGN} \leq 1\]

\[49.4\]
\[2 \leq M + 1 \leq I \leq N \land \text{MP2} \leq J_1\]

\[49.5\]
\[K_1 = I - J_1 + \text{MP1}\]

\[49.6\]
\[R(K_1, I, M, A_1, Z_1, \text{ISIGN})\]

Proof. (a) \(K_1 = I - 1\) from 44,45

(b) \(KK_1 = K_1 + 1 = I\) from 46

(c) \(S(I, M, A, Z, \text{ISIGN})\) from 39,6

(d) \(M + 1 \leq I\) and \(M \leq K_1\) from 39.3

CASE 1: \(I = M + 1, K_1 = M\)

(e) \(\gcd(A_0(1), \ldots, A_0(I)) = Z_1(M + 1)A_0(M + 1)\)

\[+ A_1(M + 1)(-2\times \text{ISIGN} + 1)A_0(M)\] from (c), 47b,48b

(f) \(R(K_1, I, M, A_1, Z_1, \text{ISIGN})\)
CASE 2: \( I > M + 1, K_1 > M \)

(g) \( A(M) = Z(I)A_o(I) + A(I)Z(I-1)A_o(I-1) \)
\[ + \sum_{k=M+1}^{I-2} \left( \sum_{j=k+1}^{I} A(j) \right) Z(k)A_o(k) \]
\[ + \left( \sum_{j=M+1}^{I} A(j) \right)(-2*ISIGN + 1)A_o(M) \text{ from (c), 39.4} \]

(h) \( \text{IGCD}_1 = Z(I)A_o(I) + A(KK_1)Z(K_1)A_o(I-1) \)
\[ + \sum_{k=M+1}^{K_1-1} \left( \sum_{j=k+1}^{K_1+1} A(j) \right) Z(k)A_o(k) \]
\[ + \left( \sum_{j=M+1}^{K_1+1} A(j) \right)(-2*ISIGN + 1)A_o(M) \text{ from (a), (b), 43} \]

(i) \( \text{IGCD}_1 = Z_1(I)A_o(I) + Z_1(K_1)A_o(I-1) \)
\[ + \sum_{k=M+1}^{K_1-1} \left( \sum_{j=k+1}^{K_1} A_1(j) \right) Z_1(k)A_o(k) \]
\[ + \left( \sum_{j=M+1}^{K_1} A_1(j) \right)(-2*ISIGN + 1)A_o(M) \text{ from 47, 48} \]

(j) \( \text{IGCD}_1 = \sum_{j=M+1}^{I} Z_1(j)A_o(j) - \sum_{j=M+1}^{K_1-1} Z_1(j)A_o(j) \)
\[ + \sum_{k=M+1}^{K_1-1} \left( \sum_{j=k+1}^{K_1} A_1(j) \right) Z_1(k)A_o(k) \]
\[ + \left( \sum_{j=M+1}^{K_1} A_1(j) \right)(-2*ISIGN + 1)A_o(M) \]

(k) \( R(K_1,I,M,A_1,Z_1,ISIGN) \)

\[ \text{IGCD}_1 = \gcd(A_o(1), \ldots, A_o(N_o)) \]

Proof. See 39.4, 39, 43, P2.
Path (39,40,41,42,43,44,45,46,47,48,49).

39  $C_1 \neq 1$
40  $I_1 = I + 1$
41  $I_1 \geq N$
42  $I_2 = N$
43  $1 \leq M \leq \text{dim}(A) \Rightarrow \text{IGCD}_1 = A(M)$
44  $J_1 = MP2$
45  $K_1 = I_2 - J_1 + MP1$
46  $KK_1 = K_i + 1$

47a  $1 \leq K_1 \leq \text{dim}(Z) \land 1 \leq KK_1 \leq \text{dim}(A) \Rightarrow Z_1(K_1) = Z(K_1) \ast A(KK_1)$
47b  $r \neq K_1 \Rightarrow Z_1(r) = Z(r)$

48a  $1 \leq K_1 \leq \text{dim}(A) \land 1 \leq KK_1 \leq \text{dim}(A) \Rightarrow A_1(K_1) = A(K_1) \ast A(KK_1)$
48b  $r \neq K_1 \Rightarrow A_1(r) = A(r)$

49.1  $1 \leq i \leq M - 1 \Rightarrow Z_1(i) = 0$
Proof. See 47b.

49.2  $I_2 + 1 \leq i \leq N_0 \Rightarrow Z_1(i) = 0$
Proof. See 42.

49.3  $MP1 = M + 1 \ast MP2 = M + 2 \ast 0 \leq \text{SIGN} \leq 1$
49.4  $2 \leq M + 1 \leq I_2 \leq N_0 \ast MP2 \ast J_1$
49.5  $K_1 = I_2 - J_1 + MP1$
49.6  $R(K_1, I_2, M, A_1, Z_1, \text{SIGN})$
Proof. See proof of 49.6 on previous path.

49.7  $\text{IGCD}_1 = \text{gcd}(A_0(1), \ldots, A_0(N_0))$
Proof. See 39.4, 39, 43, P2.
Path (49,50,45,46,47,48,49).

49 \quad J_1 = J + 1
50 \quad J_1 \leq I
45 \quad K_1 = I - J_1 + \text{MP1}
46 \quad KK_1 = K_1 + 1
47a \quad 1 \leq K_1 \leq \dim(Z) \quad 1 \leq KK_1 \leq \dim(A) \quad Z_1(K_1) = Z(K_1) * A(KK_1)
47b \quad r \neq K_1 \Rightarrow Z_1(r) = Z(r)
48a \quad 1 \leq K_1 \leq \dim(A) \quad 1 \leq KK_1 \leq \dim(A) \quad A_1(K_1) = A(K_1) * A(KK_1)
48b \quad r \neq K_1 \Rightarrow A_1(r) = A(r)

49.1' \quad 1 \leq i \leq M-1 \Rightarrow Z_1(i) = 0
Proof. See 47b.

49.2' \quad I + 1 \leq i \leq N_o \Rightarrow Z_1(i) = 0
Proof. See 47b.

49.3' \quad \text{MP1} = M + 1 \land \text{MP2} = M + 2 \land 0 \leq \text{ISIGN} \leq 1
49.4' \quad 2 \leq M + 1 \leq I \leq N_o \land \text{MP2} \leq J_1
49.5' \quad K_1 = I - J_1 + \text{MP1}
49.6' \quad R(K_1, I, M, A, Z, ISIGN)
Proof. (a) $R(K, I, M, A, Z, ISIGN)$ from 49.6
(b) $M + 1 \leq K - 1 \leq I$
(c) $\text{IGCD} = \sum_{j=k}^I Z(j)A_o(j) + \sum_{j=M+1}^K \left( \prod_{k=j+1}^K A(k) \right) Z(j)A_o(j) + \left( \prod_{k=M+2}^K A(k) \right) (-2*\text{ISIGN} + 1)A_o(M)A(M + 1)$ from (a), (b).
(d) $K_1 = K - 1$ from 49.5, 45, 49
(e) $\text{IGCD} = \sum_{j=K}^I Z_1(j)A_o(j) + A(K)Z(K-1)A_o(K-1)$

\quad + \sum_{j=M+1}^{K-2} \left( \prod_{k=j+1}^K A(k) \right) Z(j)A_o(j) + \left( \prod_{k=M+2}^K A(k) \right) (-2*\text{ISIGN} + 1)A_o(M)A(M + 1)$ from (c), (d)
(f) \[ \text{IGCD} = \sum_{j=K_1+1}^{I} Z_1(j)A_o(j) + Z_1(K_1)A_o(K_1) \]
+ \[ \sum_{j=M+1}^{K_1-1} \left( \prod_{k=j+1}^{K_1} A_1(k) \right) Z_1(j)A_o(j) \]
+ \[ \left( \prod_{k=M+2}^{K_1} A_1(k) \right) (-2*\text{ISIGN} + 1)A_o(M)A(M+1) \]
from (d),47,48

(g) \[ \text{IGCD} = \sum_{j=M+1}^{I} Z_1(j)A_o(j) - \sum_{j=M+1}^{K_1-1} Z_1(j)A_o(j) \]
+ \[ \sum_{j=M+1}^{K_1-1} \left( \prod_{k=j+1}^{K_1} A_1(k) \right) Z_1(j)A_o(j) \]
+ \[ \left( \prod_{k=M+2}^{K_1} A_1(k) \right) (-2*\text{ISIGN} + 1)A_o(M)A(M+1) \]

(h) \[ R(K_1,I,M,A_1,Z_1,\text{ISIGN}) \]

49.7' \[ \text{IGCD} = \gcd(A_o(1), \ldots, A_o(N_o)) \]
Path \((49,50,51,52,53,61)\).

\(49\)
\[ J_1 = J + 1 \]

\(50\)
\[ J_1 > I \]

\(51a\)
\[ 1 \leq M \leq \text{dim}(Z) \land 1 \leq \text{MP1} \leq \text{dim}(A) \supseteq Z_1(M) = A(\text{MP1}) \]
\[ b \]
\[ r \neq M \supseteq Z_1(r) = Z(r) \]

\(52\)
\[ \text{ISIGN} \neq 0 \]

\(53a\)
\[ 1 \leq M \leq \text{dim}(Z) \supseteq Z_2(M) = -Z_1(M) \]
\[ b \]
\[ r \neq M \supseteq Z_2(r) = Z_1(r) \]

\[61.1\] \[ \text{IGCD} = \gcd(A_0(1), \ldots, A_0(N_0)) \]

Proof. See 49.7.

\[61.2\] \[ \text{IGCD} = \sum_{i=1}^{N_0} A_0(i)Z_2(i) \]

Proof. (a) \(R(K, I, M, A, Z, \text{ISIGN})\) from 49.6

(b) \(K \leq M + 1\) from 49.3, 49.5, 49.50

(c) \[ \text{IGCD} = \sum_{j=M+1}^{I} Z_2(j)A_0(j) - A_0(M)A(M + 1) \] from (a), 49.3, 51b, 52, 53b.

(d) \[ \text{IGCD} = \sum_{j=1}^{M-1} Z_2(j)A_0(j) - A(\text{MP1})A_0(M) + \sum_{j=M+1}^{I} Z_2(j)A_0(j) \]
\[ + \sum_{j=I+1}^{N_0} Z_2(j)A_0(j) \] since first and last terms are zero, 49.1, 49.2

(e) \[ \text{IGCD} = \sum_{j=1}^{N_0} A_0(j)Z_2(j) \] from 51a, 53a.
Path \((49,50,51,52,61)\)

49 \[ J_1 = J + 1 \]

50 \[ J_1 > I \]

51a \[ 1 \leq M \leq \text{dim}(Z) \land 1 \leq MP1 \leq \text{dim}(A) \Rightarrow Z_1(M) = A(MP1) \]

b \[ r \neq M \Rightarrow Z_1(r) = Z(r) \]

52 \[ \text{ISIGN} = 0 \]

\[\begin{align*}
61.1 & \quad \text{IGCD} = \gcd(A_o(1), \ldots, A_o(N_o)) \\
\text{Proof.} & \quad \text{See 49.7.} \\
61.2 & \quad \text{IGCD} = \sum_{i=1}^{N_o} A_o(i)Z_1(i) \\
\text{Proof.} & \quad (a) \quad R(K,I,M,A,Z,\text{ISIGN}) \text{ from 49.6} \\
& \quad (b) \quad K \leq M + 1 \text{ from 49.3, 49.5, 49.50} \\
& \quad (c) \quad \text{IGCD} = \sum_{j=M+1}^{I} Z_1(j)A_o(j) + A(MP1)A_o(M) \\
& \hspace{1cm} \text{from (a), 49.3, 51b, 52.} \\
& \quad (d) \quad \text{IGCD} = \sum_{j=1}^{M-1} Z_1(j)A_o(j) + Z_1(M)A_o(M) + \sum_{j=M+1}^{I} Z_1(j)A_o(j) \\
& \hspace{1cm} + \sum_{j=I+1}^{N_o} Z_1(j)A_o(j) \text{ since first and last terms are zero, 49.1, 49.2, 51a.} \\
& \quad (e) \quad \text{IGCD} \sum_{j=1}^{N_o} Z_1(j)A_o(j)
\end{align*}\]
Path \((59,60,58,59)\).

\[ \begin{align*}
59 & \quad J_1 = J + 1 \\
60 & \quad J_1 \leq N \\
58a & \quad 1 \leq J_1 \leq \dim(Z) \implies Z_1(J_1) = 0 \\
b & \quad r \neq J_1 \implies Z_1(r) = Z(r)
\end{align*} \]

\[ \begin{align*}
59.1' & \quad 1 \leq i \leq M-1 \implies Z_1(i) = 0 \\
\text{Proof.} & \quad \text{See 59.1, 58b}
\end{align*} \]

\[ \begin{align*}
59.2' & \quad I + 1 \leq i \leq J_1 \implies Z_1(i) = 0 \\
\text{Proof.} & \quad \text{See 59.2, 58b.}
\end{align*} \]

\[ \begin{align*}
59.3' & \quad N = N_0 \land MP1 = M + 1 \land MP2 = M + 2 \land 0 \leq ISIGN \leq 1 \\
59.4' & \quad 2 \leq M + 1 \leq I \leq J_1 \leq N_0 \\
\text{Proof.} & \quad \text{See 59.4, 59, 60}
\end{align*} \]

\[ \begin{align*}
59.5' & \quad A(M) = \gcd(A_o(1), \ldots, A_o(I)) = \gcd(A_o(1), \ldots, A_o(N_0)) \\
59.6' & \quad S(I,M,A,Z_1,\text{ISIGN}) \\
\text{Proof.} & \quad \text{See 59.6, 58b.}
\end{align*} \]
Path (59, 60, 43, 44, 45, 46, 47, 48, 49).

59 \quad J_1 = J + 1
60 \quad J_1 \gg N
43 \quad 1 \leq M \leq \dim(A) \Rightarrow \gcd_1 = A(M)
44 \quad J_2 = MP2
45 \quad K_1 = I - J_2 + MP1
46 \quad KK_1 = K_1 + 1
47a \quad 1 \leq k_1 \leq \dim(Z) \land 1 \leq KK_1 \leq \dim(A) \Rightarrow Z_1(K_1) = Z(K_1) \ast A(KK_1)
   \quad r \neq K_1 \Rightarrow Z_1(r) = Z(r)
47b \quad r \neq K_1 \Rightarrow Z_1(r) = Z(r)
48a \quad 1 \leq k_1 \leq \dim(A) \land 1 \leq KK_1 \leq \dim(A) \Rightarrow A_1(K_1) = A(K_1) \ast A(KK_1)
   \quad r \neq K_1 \Rightarrow A_1(r) = A(r)
49.1 \quad 1 \leq i \leq M-1 \Rightarrow Z_1(i) = 0
   \quad \text{Proof. See 59.1, 47b.}
49.2 \quad 1 \leq i \leq N_0 \Rightarrow Z_1(i) = 0
   \quad \text{Proof. See 59.2, 47b}
49.3 \quad MP1 = M + 1 \land MP2 = M + 2 \land 0 \leq ISIGN \leq 1
49.4 \quad 2 \leq M + 1 \leq I \leq N_0 \land MP2 \leq J_2
49.5 \quad K_1 = I - J_2 + MP1
49.6 \quad R(K_1, I, M, A_1, Z_1, ISIGN)
   \quad \text{Proof. See 49.6 on Path (39, 55, 43, 44, 45, 46, 47, 48, 49).}
49.7 \quad IGCD_1 = \gcd(A_0(1), \ldots, A_0(N_0))
   \quad \text{Proof. See 59.5, 43}