MODEL CHECKING UNDER
GENERALIZED FAIRNESS CONSTRAINTS

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Abstract

In this paper, we consider whether a given finite state concurrent system meets a specification \((p_0, \phi_0)\), where \(p_0\) is a functional assertion expressed in a branching time logic, and \(\phi_0\) is the underlying fairness assumption expressed in terms of infinitary linear temporal operators \(\overline{F}p\) and \(\overline{G}p\). We will give an efficient algorithm for this problem when the fairness constraint \(\phi_0\) is an appropriate *normal form*. We will also show that the problem is, in general, NP-complete.

1. Introduction

In [CES83] an efficient algorithm is given for verifying that the global transition graph of a finite state concurrent system is a model of a specification expressed in a propositional branching time temporal logic. For finite state systems, the model checking algorithm provides a mechanizable alternative to the traditional approach to concurrent program verification where a proof of correctness is constructed by hand using various axioms and inference rules. An extension to handle certain restricted types of fairness is also considered.

In this paper, we present a model checking method to handle generalized fairness. In particular, we consider the Model Checking Problem (FMCP) for Fair Computation Tree Logic (FCTL). FCTL is a branching time system which generalizes the (ordinary) CTL of [EC82], [EH82] and [CESS3] by having all path quantifiers relativized to a fairness assumption \(\phi_0\) composed of arbitrary boolean combinations of the infinitary linear time operators \(\overline{F}p\) (*infinitely often p*) and \(\overline{G}p\) (*almost everywhere p*). Its basic modalities are thus of the form \(A_\phi\) (*for all fair paths*) or \(E_\phi\) (*for some fair path*) followed by a single linear time operator: \(Fp\) (*sometimes p*), \(Gp\) (*always p*), \(Xp\) (*nexttime p*), or \([p \cup q]\) (*p holds until q becomes true*). The infinitary operators of \(\phi_0\) make it possible to express and reason under a wide variety of *practical* fairness assumptions from the literature including weak fairness ([LA80], [PA80], [QS82]) \(\phi_0 = \bigwedge_{i=1}^{n} (\overline{F} \text{enabled}_i \Rightarrow \overline{F} \text{executed}_i)\), strong fairness ([EC80], [LA80], [PA80], [QS82]) \(\phi_0 = \bigwedge_{i=1}^{n} (\overline{F} \text{enabled}_i \Rightarrow \overline{F} \text{executed}_i)\), and impartiality [LPS81] \(\phi_0 = \bigwedge_{i=1}^{n} (\overline{F} \text{executed}_i)\). Only fairness of the last type is considered in [CES83].

We will first argue that FMCP can be efficiently reduced to the Fair State Problem (FSP): Starting from which states is there some path along which \(\phi_0\) holds? We then show that if \(\phi_0\) is of the form \(\bigwedge_{i=1}^{n} (\overline{F} p_i \lor \overline{G} q_i)\), FSP and FMCP for FCTL can be solved in time linear on the size of the input specification and input structure. Since it turns out that many practical types of fairness used in the literature can be succinctly
expressed using such a $\phi_0$, a verification method based on our model checking algorithm is potentially of wide applicability. Finally we will show that FSP and FMCP for arbitrary $\phi_0$ composed of $\exists$ p and $\forall$ p are NP-complete.

The remainder of the paper is organized as follows: Section 2 describes the syntax and semantics of our specification language -- FCTL. In section 3, the model checking problem and fair path problem are defined, and their complexity is analyzed. Applications and examples are give in section 4.

2. The Specification Language

A Fair Computation Tree Logic (FCTL) specification $(p_0, \phi_0)$ consists of a functional assertion $p_0$ and an underlying fairness assumption $\phi_0$. The functional assertion $p_0$ is expressed in essentially CTL syntax with basic modalities of the form either $A\phi$ (*for all fair paths*), or $E\phi$ (*for some fair path*) followed by one of the linear time operators $Fp$ (*sometimes p*), $Gp$ (*always p*), $Xp$ (*nexttime p*), or $[p U q]$ (*p holds until q becomes true*). We subscript the path quantifiers with the symbol $\phi$ to emphasize that they range over paths meeting the fairness constraint $\phi_0$ and to syntactically distinguish FCTL from CTL. A fairness constraint $\phi_0$ is build up from atomic propositions, the infinitary linear time operators $\exists Fp$ (*infinitely often p*) and $\exists Gp$ (*almost always p*), and boolean connectives. Note that $p_0$ is a state formula (true or false of states) whereas $\phi_0$ is a path formula (true or false of paths).

2.1. Syntax

Formally, the class of functional assertions in FCTL specifications is defined inductively as follows:

1. Any atomic proposition $P$ is a functional assertion.
2. If $p$, $q$ are functional assertions then so are $\neg p$, and $(p \land q)$.
3. If $p$, $q$ are functional assertions then so are $A_p X p$, $E_p X p$, $A_p (p U q)$, and $E_p (p U q)$.

A propositional formula is one formed by rules 1, 2 above. A fairness constraint is then formed by the following rules:

4. If $p$, $q$ are propositional formulae then $\exists F p$, $\exists G p$ are fairness constraints.
5. If $p$, $q$ are fairness constraints then so are $\neg p$, and $(p \land q)$.

The other connectives can then be defined as abbreviations in the usual way: $p \lor q$ abbreviates $\neg(\neg p \land \neg q)$, $p \Rightarrow q$ abbreviates $\neg p \lor q$, $A_p F p$ abbreviates $A_p (true U p)$,
EϕFp abbreviates Eϕ(true U p), AϕGp abbreviates ¬Eϕ¬p, etc.

Remark: Recall that CTL* is the full branching time logic in which the basic modalities have the form: A or E followed by an arbitrary combination (involving boolean connectives and nesting) of linear time operators F, G, X, and U. We could thus view the assertions of FCTL as a sublanguage of CTL* where, eg., the AϕFp is an abbreviation for the CTL* formula A[ϕ0 ⇒ Fp]. However, the corresponding CTL* formula might be rather unwieldy due to the need to repeatedly write down multiple copies of the actual fairness formula ϕ0.

2.2. Semantics

Let AP be the underlying set of atomic propositions. A structure M=(S, R, L) is a labeled transition graph, where

- S is a nonempty set of states.
- R is a binary relation on S which gives the possible transitions between states.
- L: S → 2AP, is a labeling which assigns to each state a set of atomic propositions (intuitively, the propositions true at that state).

The size of a structure M=(S, R, L), written |M|, is defined to be the sum of the number of states in S and the number of transitions in R.

A fullpath x is an infinite sequence of states (s0, s1, s2, ... ) such that ∀i≥0[(si, si+1)] ∈ R]. We use x^i to denote the suffix of x beginning at state x_i, i.e. x^i=(x_i, x_i+1, ... ). Sometimes we use "computation sequence" instead of "fullpath". We write M, x|=ϕ0 to mean that fullpath x in structure M meets fairness constraint ϕ0. We define the |= relation inductively in the usual way:

1. M, x|=p iff P ∈ L(s0), for any atomic proposition P.
2. M, x|=¬p iff not(M, x|=p)
3. M, x|=p ∧ q iff M, x|=p and M, x|=q
4. M, x|=ϕ p iff there exists infinitely many i≥0 such that M, x_i|= p
5. M, s|=Eϕp if there is a fullpath x starting at s such that M, x|=ϕ0. We say that x is a fair path in structure M under fairness assumption ϕ0 if M, x|=ϕ0 holds. A state is fair iff it lies on some fair path. A substructure C of M is called a fair component if C is strongly connected and contains some fair path.
An FCTL specification \((p_0, \phi_0)\) is interpreted with respect to a structure \(M\). We write \(M, s \models_{\phi_0} p_0\) to mean that functional assertion \(p_0\) is true at state \(s\) of structure \(M\) under fairness assumption \(\phi_0\). We define \(\models_{\phi_0}\) inductively on the structure of the functional assertion \(p_0\):

1. \(M, s \models_{\phi_0} P\) iff \(P \in L(s)\), for any atomic proposition \(P\).
2. \(M, s \models_{\phi_0} \neg p\) iff \(\not(M, s \models_{\phi_0} p)\)
3. \(M, s \models_{\phi_0} p \land q\) iff \(M, s \models_{\phi_0} p\) and \(M, s \models_{\phi_0} q\)
4. \(M, s_0 \models_{\phi_0} \exists X p\) iff there exists a path \(x = (s_0, s_1, s_2, \ldots)\) such that \(M, x \models_{\phi_0}\) and \(M, s_1 \models_{\phi_0} p\)
5. \(M, s_0 \models_{\phi_0} \forall X p\) iff for all paths \(x = (s_0, s_1, s_2, \ldots)\) \(\left[M, x \models_{\phi_0} \Rightarrow M, s_1 \models_{\phi_0} p\right]\)
6. \(M, s_0 \models_{\phi_0} \exists (p \cup q)\) iff there exists a path \(x = (s_0, s_1, s_2, \ldots)\) such that \(M, x \models_{\phi_0}\) and \(\exists j \geq 0[M, s_j \models_{\phi_0} q \land \forall i < j(M, s_i \models_{\phi_0} p)]\)
7. \(M, s_0 \models_{\phi_0} \forall (p \cup q)\) iff for all paths \(x = (s_0, s_1, s_2, \ldots)\), \(M, x \models_{\phi_0}\) implies \(\exists j \geq 0[M, s_j \models_{\phi_0} q \land \forall i < j(M, s_i \models_{\phi_0} p)]\)

We say that an FCTL specification \((p_0, \phi_0)\) is valid provided that for each structure \(M = (S, R, L)\), and each state \(s \in S\), \(M, s \models_{\phi_0} p_0\) holds. An FCTL specification \((p_0, \phi_0)\) is satisfiable provided that for some structure \(M = (S, R, L)\) and some state \(s \in S\), \(M, s \models_{\phi_0} p_0\). In the later case, we say that \(M\) is a model for \((p_0, \phi_0)\).

If \((p_0, \phi_0)\) is an FCTL specification then the set of subformulae of functional assertion \(p_0\), written \(SF(p_0)\), is the minimal set satisfying the following conditions:

1. \(p_0 \in SF(p_0)\)
2. \(\neg p \in SF(p_0) \Rightarrow p \in SF(p_0)\)
3. \(p \land q \in SF(p_0) \Rightarrow p, q \in SF(p_0)\)
4. \(\exists X p \in SF(p_0) \Rightarrow p \in SF(p_0)\)
5. \(\forall X p \in SF(p_0) \Rightarrow p \in SF(p_0)\)
6. \(\exists (p \cup q) \in SF(p_0) \Rightarrow p, q, \exists X \exists (p \cup q) \in SF(p_0)\)
7. \(\forall (p \cup q) \in SF(p_0) \Rightarrow p, q, \forall X \forall (p \cup q) \in SF(p_0)\)

Note that \(|SF(p_0)| = O(|p_0|)\), where \(|SF(p_0)|\) denotes the number of elements in \(SF(p_0)\), and \(|p_0|\) denotes the length of the formula \(p_0\).
3. Model Checking Problem and Fair State Problem

The Model Checking Problem for FCTL (FMCP) is: Given a structure $M=(S, R, L)$, and an FCTL specification $(p_0, \phi_0)$, determine for each state $s \in S$ whether $M, s \models \phi_0 \land p_0$. The Fair State Problem (FSP) is: Given a structure $M=(S, R, L)$, and a fairness constraint $\phi_0$, determine for each state $s \in S$ whether there is a fullpath $x$ in $M$ starting at $s$ such that $M, x \models \phi_0$.

3.1. Reduction of FMCP to FSP

Since the FSP condition is equivalent to $M, s \models \phi_0 \land E_x X_{true}$, FSP may be viewed as a special case of FMCP. However, we can generalize a method in [CES83] to reduce FMCP to FSP. The reduction yields an algorithm for FMCP that runs in time linear in the size of the input (specification and structure) and the time to solve FSP. The reduction exploits the observation that, for any fairness constraint $\phi_0$ and for any fullpaths $x$ and $y$ such that $x$ is a suffix of $y$, $M, x \models \phi_0$ iff $M, y \models \phi_0$. We thus get the following equivalences:

1. $M, s \models \phi_0 \land E_x X_{true}$ iff $\exists (s, t) \in R [(M, t \models E\phi_0) \land (M, t \models \phi_0 \land p)]$
2. $M, s \models \phi_0 \land A_x X_{true}$ iff $\forall (s, t) \in R [(M, t \models E\phi_0) \Rightarrow (M, t \models \phi_0 \land p)]$
3. $M, s \models \phi_0 \land E_x [p \land q]$ iff there is a finite path of nodes satisfying $p$ leading to a fair node which satisfies $q$.
4. $M, s \models \phi_0 \land A_x [p \land q]$ iff $M, s \models \phi_0 \land E_x [p \land q]$ and $E_x [\neg q \land (\neg p \land \neg q)] \lor E_x G(\neg q)$

The equivalences (1) and (2) are immediate; (3) is just a restatement of the definition. To check $A_x [p \land q]$, equivalence (4) shows that we can first check whether $E_x [\neg q \land (\neg p \land \neg q)]$ using equivalence (3). To next check $E_x G(\neg q)$, we let $M'$ be the substructure of $M$ obtained by deleting all nodes which $q$ holds (inductively, we assume nodes of $M$ are labeled with the true subformulae). Then $E_x G(\neg q)$ holds at a node $s$ iff there is a finite path from $s$ to a fair node $t$ in $M'$. Detection of fair nodes is done by the algorithm for FSP. The reduction algorithm (call it AFMCP) is described in greater detail below (Figure 1).

We claim that when AFMCP terminates, the following assertion holds: $\forall f \in SF(p_0)[f \in \text{label}(s) \iff M, s \models \phi_0 f]$. The proof is by induction on the structure of $f$. The details are left to the reader.
Let $T_A(M, \phi_0)$ denote the time complexity of algorithm AFSP($M, \phi_0$). Then we calculate that AFMCP requires time $O(|p_0|\cdot \text{max}(|M|, T_A(M, \phi_0)))$. To see this note that steps 1, 2 are executed only once and require time $O(T_A(M, \phi_0)+O|M|)$. Moreover, step 3 is a for loop which is executed $|SF(p_0)|$ times and each case in the loop requires time $O(|M|)$ except the case where $f=\exists \phi[p \lor q]$ which requires time $O(|M|)+T_A(M, \phi_0)$. Hence, step 3 requires time less than $|SF(p_0)|\cdot \text{max}(O(|M|), T_A(M, \phi_0))=O(|p_0|\cdot \text{max}(|M|, T_A(M, \phi_0)))$, so does the whole algorithm.

### 3.2. Efficient Algorithm for Fair State Problem

In many interesting cases, FSP (and hence FMCP) can be solved in time linear in the size of the input structure and specification. We now consider a normal form for these cases: $\phi_0=\bigwedge_{i=1}^{n} (\bigvee_{j=1}^{k} p_i \lor \bigvee_{j=1}^{k} q_i)$. Since it turns out that almost all practical fairness notions can be succinctly expressed by a $\phi_0$ of the form, we believe that FCTL restricted to such fairness constraints is still of great applicability. We will develop the linear time algorithm for FMCP under such a restriction in the coming paragraphs.

We first show how detect fair components (recall that a strongly connected structure is fair iff there is a fair path in it). Given a strongly connected structure $C=(S, R, L)$, and a fairness constraint $\phi_0=\bigwedge_{i=1}^{k} (\bigvee_{j=1}^{k} p_i \lor \bigvee_{j=1}^{k} q_i)$, the algorithm described in Figure 2 can decide whether $C$ is a fair component w.r.t. $\phi_0$ in time $O(|C|\cdot |\phi_0|)$.

The proof that $C$ is a fair component w. r. t. $\phi_0$ iff the recursive function AFC($C$, $\phi_0$) returns true is by induction on the number of the conjuncts $k$ in $\phi_0$.

**Basis:** $k=0$, $\phi_0=$true, and the program AFC returns true immediately. Hence the hypothesis holds. (Note that any strongly connected component is fair w.r.t. *true*.)

**Induction step:** Assume true for $k<n$, prove for $k=n$ as follows:

[Only if part]: If AFC returns true, then it must do so either at statement (6) or statement (8). Case 1: AFC returns true at statement (6). By induction hypothesis, at least one of the strongly components in $C'$ is fair w.r.t. $\phi'_0$. Let $D$ be one of such strongly connected components. Since $D$ is contained in $C'$ and every nodes of $C'$ satisfies $q_i$, every path in $D$ satisfies $\exists q_i$. Hence $D$ is also a fair component w.r.t. to $\phi_0$. Hence $C$ is a fair component.

Case 2: AFC returns true at statement (8). In this case, some node in $C$ satisfies $p_i \forall i \in [1,n]$. Hence any cycle in $C$ which includes all nodes of $C$ is a fair path w.r.t. $\phi_0$. 

(because C is strongly connected, there exists at least one such path). Let x be one such cycle; it's obvious that M, x|\models \phi_0. Hence C is a fair component.

[If part]: Assume that C is a fair component, we prove that AFC will return true either at statement (6) or at statement (8). (The following argument is essentially the reverse of the previous proof.)

Case 1: \( \forall j \in [1, n](\exists s \in S(C, s|\models \phi_0 p_j)) \). In this case the condition of statement (5) is always false. Hence the program will terminate at statement (8).

Case 2: \( \exists j \in [1, n](\forall s \in S(\text{not } C, s|\models \phi_0 p_j)) \). Let i be the smallest integer such that \( \forall s \in S(\text{not } C, s|\models \phi_0 p_j) \). Since C is fair, \( C \) contains some fair cycle x w.r.t. \( \phi_0 \). Every node on x must satisfy \( q_i \). Hence x must be included in some strongly connected component \( D \) of \( C' \). By induction hypothesis, AFC(D, \( \phi_0' \)) will return true, and so will FC(C, \( \phi_0 \)).

To analyze the complexity, let \( m = |C|, n = |\phi_0| \), and \( k = \) the number of of conjuncts of \( \phi_0 \). Define function \( T(i, j) \) to be the time complexity of AFC when the size of input structure is i and the input fairness constraint has k conjuncts. Let \( X = \{D_1, \ldots, D_k\} \) be the set of strongly connected components of \( C' \). If we let \( d_i \) denote \( |D_i| \), then \( \Sigma d_i \leq |C'| \leq m \). Clearly, \( T(m, 0) = O(1) \) since the program AFC returns true immediately. Note that for any recursive call each statement in AFC can be executed at most \( k \) times. Furthermore, the compound statement beginning at (5) can be executed at most once (because it always returns control to the caller). Hence we have the following recurrence relation:

\[
T(m, k) \leq \sum_{i=1}^{k} O(m \cdot |p_i|) + \sum_{i=1}^{k} T(d_i, k-1)
\]

Which can be simplified to \( T(m, k) \leq O(m \cdot n) + \sum_{i=1}^{k} T(d_i, k-1) \). By induction on \( k \), we can easily show that \( T(m, k) \leq O(m \cdot n) \). The details are left to the reader.

The program AFS(M, \( \phi_0, S' \)) of Figure 3 is an algorithm for FSP of time complexity \( O(|M| \cdot |\phi_0|) \). The for-loop checks fair components in M, if a state \( s \) is in a fair component then it is a fair state. The while-loop picks out nodes which can reach a fair state. It is clear that a state which can reach another fair state is a fair state. So the algorithm picks fair states. Conversely, for any fair state \( s \) in M, if \( s \) lives on a fair component then it will be put in \( S' \) during the execution of the for-loop; otherwise, \( s \) can reach a fair state and will be put into \( S' \) during the execution of the while-loop.

The complexity bound follows from the complexity analysis of algorithm AFC. To
see this, assume that $M = (S, R, L)$ can be partitioned into $l$ strongly connected components $C_1, C_2, \ldots, C_l$. Then each step of the for-loop requires time $|AFC(C_i, \phi_0)| + O(|C_i|)$ which is equal to $O(|C_i| \cdot |\phi_0|)$. Hence the for-loop requires time $O(|M| \cdot |\phi_0|)$. The while loop requires only $O(|M|)$ time, so the whole algorithm takes only $O(|M| \cdot |\phi_0|)$ time.

We have thus established,

**Theorem 1:** FMCP for input structure $M = (S, R, L)$, and input specification $(p_0, \phi_0)$ with $\phi_0 = \bigwedge_{i=1}^n (\bigvee_i^{\infty} p_i \lor \bigvee_i^{\infty} q_i)$ can be solved in time $O(|p_0| \cdot |M| \cdot |\phi_0|)$.

3.3. Complexity of The General Case

We show, in this section, that FSP (and hence also FMCP) is NP-complete for general fairness specification $\phi_0$.

**Theorem 2:** FSP is NP-complete.

Proof: We will reduce 3-SAT to FSP, with fairness constraint of the form $\bigwedge_{i=1}^n (\bigvee_i^{\infty} p_i \lor \bigvee_i^{\infty} q_i)$.

Given a formula $g$ in 3-CNF with $n$ variables and $m$ factors, we show how to construct, in polynomial time, a structure $M = (S, R, L)$ with a designated state $s \in S$, and a fairness constraint $\phi_0$ such that there is a path $z$ in $M$ starting from $s$ and $M$, $z \models \phi_0$ iff $g$ is satisfiable.

Let $x_1, x_2, \ldots, x_n$ and $C_1, C_2, \ldots, C_m$ be the variables and factors of $g$ (i.e. $g = \bigwedge_{i=1}^m C_i$), where $C_i = l_{i1} \lor l_{i2} \lor l_{i3}$ for $1 \leq i \leq m$, and $l_{ij} = x_k$ or $\neg x_k$ for some $k \in [1, n]$. Take $AP = \{p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n\}$ as the underlying set of atomic propositions. Construct a structure $M = (S, R, L)$ as follows:

$S = \{s, t\} \cup \{v_{ij} : 1 \leq i \leq m, 1 \leq j \leq m\}$

$R = \{(s, v_{ij}) : 1 \leq j \leq 3\} \cup \{(v_{mj}, t) : 1 \leq j \leq 3\} \cup \{(t, s)\}$

$\quad \cup \{(v_{ij}, v_{i+1,k}) : 1 \leq i \leq m-1 \text{ and } 1 \leq j, k \leq 3\}$

$L(s) = L(t) = \emptyset$

$L(v_{ij}) = \{p_k\}$ if $l_{ij} = x_k$

$\quad = \{q_k\}$ if $l_{ij} = \neg x_k$

Let $\phi_0 = \bigwedge_{i=1}^n (G \neg p_i \lor G \neg q_i)$.

Structure $M$ is shown in Figure 4. It is quite clear that the above construction can be
done in polynomial time. We claim that \( g \) is satisfiable iff there is some path \( z \) in \( M \) starting from \( s \) such that \( M, z \models \phi_0 \). Proof of the claim is given in the appendix.

[Membership]: It has already been shown in [SC82] that the model checking problem for linear time temporal logic with \( F \), and \( G \) operators can be solved in NP time. Hence \( FSP \) is in NP, thus \( FSP \) is NP-complete. In [SC82] it was shown that, in effect, \( FSP \) for any arbitrary linear time formula over \( F, G \) is NP-complete. For \( FSP \) with \( \phi_0 \) of the type we construct, membership in NP follows since our language of fairness constraints may be viewed as a sublanguage of linear time logic by the equivalences \( \varphi \ 
abla p \equiv GFp \) and \( \varphi \ 
abla p \equiv FGp \). But NP-hardness for \( \phi_0 \) of our type does not follow from the proof in [sc82] which used a different reduction to a formula \( Fp_1 \land \ldots \land Fp_n \). Because \( Fp \) is not expressible in our \( \phi_0 \) language, that proof cannot be applied. Since our \( \phi_0 \) language has a more restricted syntax, its decision problem might be easier. Our NP-hardness argument shows that such is not the case: even a simple \( \phi_0 \) of the form \( \varphi \ 
abla p \lor \varphi \ 
abla q \) can be "hard". This is surprising since the dual property \( \varphi \ 
abla p \land \varphi \ 
abla q \) is "easy".

Corollary: FMCP is NP-complete.

Proof: NP-hardness follows directly from Theorem 2, and NP-membership follows from the reduction algorithm in Figure 1 and the Theorem 2.

4. Applications

Our canonical form fairness specification \( \phi_0 = \land_{i=1}^n (Gp_i \lor Fq_i) \) allows us to perform model checking efficiently under many commonly used "practical" fairness assumptions and a wide variety of fairness related notions. To handle a given fairness-type notion using the canonical \( \phi_0 \) form, it is often helpful to apply the following equivalences:

1. \( E(p \lor q) \equiv Ep \lor Eq \)
2. \( \varphi \ 
abla (p \lor q) \equiv \varphi \ 
abla p \lor \varphi \ 
abla q \)
3. \( \varphi \ 
abla (p \land q) \equiv \varphi \ 
abla p \land \varphi \ 
abla q \)

Note that a consequence of (1) is that any \( \phi_0 \) of the form \( \lor_{i=1}^n \land_{j=1}^m (Gp_{ij} \lor Fq_{ij}) \) can be handled with linear time complexity as well because \( E\phi_0 \equiv \lor_{i=1}^n \land_{j=1}^m (Gp_{ij} \lor Fq_{ij}) \). To see if a node is fair w.r.t. \( \phi_0 \), one merely checks if it is fair w.r.t. one of the disjuncts of \( \phi_0 \). Such a \( \phi_0 \) may thus viewed as a generalized canonical form for doing model checking.
in linear time.

4.1. Expressible Fairness Notions

We can succinctly express the following fairness notions using our generalized canonical form (with a liberal interpretation of the meanings of atomic propositions):

1. Impartiality: \( \bigwedge_{i=1}^{n} (\exists^{\infty} \text{executed}_i) \)

2. Weak Fairness: \( \bigwedge_{i=1}^{n} (\neg \text{enabled}_i \Rightarrow \exists^{\infty} \text{executed}_i) \)

   \( \equiv \bigwedge_{i=1}^{n} (\exists^{\infty} (\neg \text{enabled}_i \lor \exists^{\infty} \text{executed}_i)) \)

   \( \equiv \bigwedge_{i=1}^{n} (\exists^{\infty} (\neg \text{enabled}_i \lor \exists^{\infty} \text{executed}_i)) \)

3. Strong Fairness: \( \bigwedge_{i=1}^{n} (\exists^{\infty} \text{enabled}_i \Rightarrow \exists^{\infty} \text{executed}_i) \equiv \bigwedge_{i=1}^{n} (\neg \text{enabled}_i \lor \exists^{\infty} \text{executed}_i) \)

The following definitions of acceptance for finite automata on infinite strings are closely related to fairness and can also be readily expressed:

4. Buchi Acceptance: \( \exists^{\infty} \text{Green} \)

5. Pairs Acceptance: \( \bigvee_{i=1}^{n} (\neg \exists^{\infty} \text{Red}_i \land \exists^{\infty} \text{Green}_i) \equiv \bigvee_{i=1}^{n} (\neg \exists^{\infty} \text{Red}_i \land \exists^{\infty} \text{Green}_i) \)

6. Complemented Pairs Acceptances: \( \bigwedge_{i=1}^{n} (\exists^{\infty} \text{Red}_i \Rightarrow \exists^{\infty} \text{Green}_i) \)

   \( \equiv \bigwedge_{i=1}^{n} (\neg \exists^{\infty} \text{Red}_i \lor \exists^{\infty} \text{Green}_i) \)

4.2. Extended Semantics with Arc Labels

There was a point glossed over in our rendering of the fairness properties above, whereas the enabling condition for performing a step of process \( i \) is properly viewed as a predicate on states (i.e. nodes), the actual execution of the step is more naturally modeled as a transition (i.e. traversal of an arc). To allow a precise differentiation between execution of transition actions and enabling of state conditions, we can extend the semantics of FCTL so that a structure \( M = (S, A_1, A_2, \ldots, A_p, L) \) where each \( A_i \subseteq S \times S \) represents (the atomic actions of) process \( i \), and where we think of each arc \( (s_1, s_2) \in A = A_1 \cup \ldots \cup A_p \) as being labeled with the set \( \{ i : (s_1, s_2) \in A_i \} \) of processes which can cause a transition from state \( s_1 \) to state \( s_2 \). We can now extend the fairness specifications to allow atomic assertions: \( \text{executed}_i \) hold at \( (s_1, s_2) \) iff \( (s_1, s_2) \in A_i \). The fairness
specifications such as $\overline{\mathcal{R}}_{\text{enabled}_i} \Rightarrow \overline{\mathcal{R}}_{\text{executed}_i}$ can be given a rigorous definition. It is straightforward to formalize this approach and to extend our efficient model checking algorithm to the extended semantics, but the details are tedious.

As an alternative, we can encode the extended semantics with arc labels into the original semantic framework. One way to do this involves duplication of nodes: Given an extended structure $M = (S, A_1, \ldots, A_p, L)$ we can form an ordinary structure $M' = (S', A', L')$ where $S' = \{(s, i) \in S \times [0:p] : s \in S$ and $[(i = 0$ and $s$ has no predecessor in $M$) or $(i \in [1:p]$ and $s$ is an $A_i$-successor for some state $t$ in $M)]\}$, $A' = \{((s, i), (t, j)) \in S' \times S' : (s, t) \in A_j\}$, and for $i \in [1:p]$, $L'(s, i) = L(s) \cup \{\text{executed}_i\}$ where $\text{executed}_i$ is a distinguished proposition while for $i = 0$, $L'(s, 0) = L(s)$. Intuitively, $(s, i)$ is a copy of state $s$ reached by a transition of process $i$. See Figure 5. Also note that one potential drawback of this method is that the size of the graph representation of the new structure can be $p^2$ times the size of the old structure, where $p$ is the number of processes. This method is similar to that of [Pn77] and [CES83].

Another way to encode the extended semantics into the original framework is to insert an intermediate node "in the center" of each arc. Each such inserted node is labeled with a distinguished proposition $\text{Arc}$ plus the process numbers labeling its corresponding arc. The label of each original node is augmented with a distinguished proposition $\text{Node}$. Formally, if $M = (S, A_1, \ldots, A_p, L)$ then the new structure $M' = (S', A', L')$ where, letting $A = A_1 \cup \ldots \cup A_p$, $S' = S \cup A$, $A' = \{(s, t) : s \in S$ and $(s, t) \in A\} \cup \{((s, t), t) : (s, t) \in A$ and $t \in S\}$, $L'(s) = L(s) \cup \{\text{Node}\}$ for $s \in S$, and $L'(t, i) = \{\text{Arc}\} \cup \{i \in [1:p] : (s, t) \in A_i\}$

Let $\dagger$: assertion of $M \rightarrow$ assertion of $M'$ be a translation defined recursively as follows:

1. $P^\dagger = P$ where $P$ is an atomic proposition
2. $(\neg p)^\dagger = \neg p^\dagger$
3. $(p \land q)^\dagger = p^\dagger \land q^\dagger$
4. $(E\phi Xp)^\dagger = E\phi [X E\phi Xp]^\dagger$
5. $(A\phi Xp)^\dagger = A\phi [X A\phi Xp]^\dagger$
6. $(E\phi [p \cup q])^\dagger = E\phi [(\text{Node} \Rightarrow \text{p}) \cup (\text{Node} \land q)]$

We then rewrite the original specification according to the translation $\dagger$. The method is illustrated in Figure 6. One advantage of this encoding is that the size of (the graph representing) the new structure does not depending on the number of processes; rather, the increase in size is linear. If the graph representing the original structure has $v$ nodes and $e$ arcs, the new structure's graph has $v + e$ nodes and $2e$ arcs. Of course, there is also a linear blowup in the size of the specification formula.
4.3. Examples

Example 1 (Dijkstra’s random number generator): Figure 7.(a) is a guarded command program for a random number generator, the corresponding global state graph is given in Figure 7.(b), and Figure 7.(c) is the resulting graph of 7.(b) after applying the first encoding scheme discussed in the previous section. We want to show that under strong fair scheduling of processes, the program fairly terminates, i.e. the program terminates on every path satisfying the strong fair scheduling condition. We can express the above statement as an FCTL specification: \((p_o, \phi_0) \models (A \phi F A \phi X false, (\text{\textit{enabled}}_1 \Rightarrow \text{\textit{executed}}_1) \land (\text{\textit{enabled}}_2 \Rightarrow \text{\textit{executed}}_2), \) where \(\phi_0\) is the fairness assumption which requires that if a process is enabled infinitely often then it must be executed infinitely often, and \(p_o\) is the functional assertion which asserts that along every fair path the program terminates. We then apply our model checking algorithm to \((p_o, \phi_0)\) on the model in Figure 7.(c). The result is given in Figure 7.(d) which shows that \(p_o\) holds on every state of the structure. Hence the program in Figure 7.(a) fairly terminates.

Note that if we change the underlying fairness assumption in the above example to either weak fairness or impartiality, program 7.(a) still terminates fairly because the only infinite execution sequence is unfair w.r.t. all these three notions of fairness. However, if no fairness assumption is taken, the program might always execute process 1 and never terminates.

Example 2 (Mutual Exclusion): We illustrate our efficient model checking algorithm by considering a solution to the mutual exclusion problem for two processes \(P_1\) and \(P_2\). In the solution each process is always in exactly one of the three code regions:

- \(N_i\): the Noncritical region.
- \(T_i\): the Trying region.
- \(C_i\): the Critical region.

A global state transition graph is given in Figure 8. Note that we only record transitions between different regions of code; internal moves within the same region are not considered.

To establish absence of starvation, we must show that \(T_i \Rightarrow A \phi FC_i\) for each process \(i\). Note that the solution is not starvation free under an unfair scheduler, for example, in the infinite execution sequence \(s_0; s_1; s_4; s_7; s_1; s_4; s_7; \ldots\) process 1 is in its trying region infinitely often but it never enters its critical region. We will show that the solution is starvation free under the weak fairness assumption \(\phi_0 = (\text{\textit{enabled}}_1 \Rightarrow \text{\textit{executed}}_1) \land (\text{\textit{enabled}}_2 \Rightarrow \text{\textit{executed}}_2)\). Without loss of generality, we only consider the starvation
free property for process 1: \( p_0 = A_\phi G(\neg T_1 \vee A_\phi FC_1) \). The states of the global transition graph will be labeled with subformulae of \( p_0 \) during execution of model checking algorithm. On termination every state will be labeled with \( \neg T_1 \vee A_\phi FC_1 \), as shown in Figure 8.(b). Thus we can conclude that \( s_0 \models \phi_0 p_0 \). It follows that process 1 cannot be prevented from entering its critical region once it has entered its trying region.

Appendix: Proof of Theorem 1 Continued

[only if part]: Assume that \( g \) is satisfiable. Since \( g \) is satisfiable, there exists a truth assignment \( A \) such that \( g \) is true under \( A \), i.e. for any factor \( C_i \), there is a literal \( l_{ij} \) in \( C_i \) such that \( l_{ij} \) is true under this particular truth assignment. Now consider a cycle \( z \) in \( M \) formed by nodes \( s, v_{1j_1}, ... v_{nj_n}, t \), such that for all \( i, v_{ij_i} \) is true under the assignment \( A \).

We will show that \( M, z \models \phi_0 \) by showing that \( \overset{\infty}{G} \neg p_k \vee \overset{\infty}{G} \neg q_k \) holds on \( z \) for every \( k \in [1,n] \). If \( \overset{\infty}{G} \neg p_k \) holds on \( z \) then \( \overset{\infty}{G} \neg p_k \vee \overset{\infty}{G} q_k \) also holds on \( z \). Hence, we only have to show that when \( \overset{\infty}{G} \neg p_k \) does not hold on \( z \), \( \overset{\infty}{G} \neg p_k \vee \overset{\infty}{G} \neg q_k \) still holds on \( z \). Because \( \overset{\infty}{G} \neg p_k \) does not hold on \( z \), there must be some node \( v \) in \( z \) such that \( p_k \in L(v) \). Note that \( L(s) = L(t) = \emptyset \). Hence \( v \) is \( v_{ij_i} \) for some \( i \in [1,n] \). By the construction of the labeling function \( L \), we conclude that \( l_{ij_i} \models x_k \). By the construction of \( z \), \( l_{ij_i} \) is true, i.e. the assignment \( A \) assigns true to \( x_k \). Hence \( \neg x_k \) is false under \( A \). Clearly under \( A \), \( l_{ij_i} \not\models \neg x_k \) for any \( i \in [1,n] \). Again by the definition of \( L \), \( q_k \not\models l_{ij_i} \), i.e. \( \overset{\infty}{G} \neg q_k \) holds on \( z \). Hence \( \overset{\infty}{G} \neg p_k \vee \overset{\infty}{G} \neg q_k \) holds on \( z \) for any \( k \in [1,n] \). We conclude that \( \phi_0 \) holds on \( z \).

[if part]: Assume that there is a path \( z \) in \( M \) starting from \( s \) such that \( \phi_0 \) holds on \( z \). Let \( z' \) be a suffix of \( z \) starting from state \( s \) such that \( \bigwedge_{i=1}^{n} (G \neg p_k \vee G \neg q_k) \) holds on \( z' \). Note that either \( p_k \) or \( q_k \) does not appear on the label of any node on \( z' \). Consider the truth assignment \( A : x_k \rightarrow \{T, F\} \) as follows:

\[
A(x_k) = T \quad \text{if } \exists i,j[p_k \in L(v_{ij})] \\
A(x_k) = F \quad \text{if } \exists i,j[q_k \in L(v_{ij})]
\]

It is quite easy to check that \( A \) is consistent in the sense that \( A \) assigns a unique value to each \( x_k \). Furthermore, the assignment caused by any \( L(v_{ij}) \) will guarantee that \( C_i \) is true under the assignment \( A \). Hence \( g \) is satisfiable. This completes our proof.
REFERENCES:


Let \( \text{AFSP}(M, \Phi_0) \) be an algorithm for solving FSP which returns the set of fair states of the input structure \( M \) w.r.t. fairness constraint \( \Phi_0 \).

---

**procedure** \( \text{AFMCP}(M, (p_0, \Phi_0)) \):

\[
\begin{align*}
&\text{/* } M = (S, R, L) \text{ is the input structure, and } (p_0, \Phi_0) \text{ is the specification */} \\
&\begin{align*}
&\text{begin} \\
&1. \quad S' := \text{AFSP}(M, (p_0, \Phi_0)) \quad \text{/* use algorithm AFSP to identify fair states in } M \text{ */} \\
&2. \quad \text{for each } s \in S \text{ do if } s \in S' \text{ then } \text{label}(s) := \{\Phi_0\} \text{ else } \text{label}(s) := \{\neg\Phi_0\}; \\
&3. \quad \text{for each formula } \varphi \in \text{SF}(p_0) \text{ do /* Inductively, taking the shortest formula first. */} \\
&\quad \text{case } \varphi \text{ of the form} \\
&\quad 3.1 \quad \text{atomic formula: for each } s \in S \text{ do if } s \in L(s) \text{ then } \text{label}(s) := \text{label}(s) \cup \{f\}; \\
&\quad 3.2 \quad \neg p; \text{ for each } s \in S \text{ do if } s \notin \text{label}(s) \text{ then } \text{label}(s) := \text{label}(s) \cup \{f\}; \\
&\quad 3.3 \quad p \land q; \text{ for each } s \in S \text{ do if } p, q \in \text{label}(s) \text{ then } \text{label}(s) := \text{label}(s) \cup \{f\}; \\
&\quad 3.4 \quad \text{E}_\varphi Xp; \text{ for each } s \in S \text{ do if } \exists (s, t) \in R[p, \Phi_0 \in \text{label}(t)] \\
&\quad \quad \text{then } \text{label}(s) := \text{label}(s) \cup \{f\}; \\
&\quad 3.5 \quad \text{A}_\varphi Xp; \text{ for each } s \in S \text{ do if } \forall (s, t) \in R[\Phi_0 \in \text{label}(s) \Rightarrow p \in \text{label}(s)] \\
&\quad \quad \text{then } \text{label}(s) := \text{label}(s) \cup \{f\}; \\
&\quad 3.6 \quad \text{E}_\varphi(p \cup q); \\
&\quad \quad \text{EU} := \text{empty set}; \\
&\quad \quad \text{for each } s \in S \text{ do} \\
&\quad \quad \quad \text{if } q, \Phi_0 \in \text{label}(s) \text{ then begin } \text{label}(s) := \text{label}(s) \cup \{f\} \\
&\quad \quad \quad \quad \text{EU} := \text{EU} \cup \{s\} \\
&\quad \quad \quad \text{end;} \\
&\quad \text{while } \text{EU} \neq \emptyset \text{ do} \\
&\quad \quad \text{begin} \\
&\quad \quad \quad \text{let } t \text{ be the first element of } \text{EU}; \\
&\quad \quad \quad \text{D} := \{s \in S: (s, t) \in R \land p \in \text{label}(s) \land q, \text{E}_\varphi(p \cup q) \in \text{label}(s)\}; \\
&\quad \quad \quad \text{for each } s \in D \text{ do } \text{label}(s) := \text{label}(s) \cup \{f\}; \\
&\quad \quad \quad \text{EU} := \text{EU} \cup \text{D} \setminus \{t\}; \\
&\quad \quad \text{end of while;} \\
&\quad 3.7 \quad \text{A}_\varphi(p \cup q); \\
&\quad \quad \text{Label the states of } M \text{ with } \neg p, \neg q, \neg p \land \neg q \text{ by using 3.2 and 3.3.} \\
&\quad \quad \text{Label the states of } M \text{ with } \text{E}_\varphi\neg q \cup (\neg p \land \neg q) \text{ using 3.6.} \\
&\quad \quad \text{Label the states of } M \text{ with } \neg \text{E}_\varphi\neg q \cup (\neg p \land \neg q) \text{ using 3.2.} \\
&\quad \quad \text{S'} := \{s \in S: \neg q \in \text{label}(s)\}; \\
&\quad \quad \text{M'} := (S', R[S \times S'], L[S']); \\
&\quad \quad \text{FS'} := A(M', \Phi_0); \\
&\quad \quad \text{for all } s \in FS' \text{ do } \text{label}(s) := \text{label}(s) \cup \{\text{E}_\varphi G\neg q\}; \\
&\quad \quad \text{Label the states of } M \text{ with } \neg \text{E}_\varphi G\neg q; \\
&\quad \quad \text{for all } s \in S \text{ do} \\
&\quad \quad \quad \text{if } \neg \text{E}_\varphi G\neg q, \neg \text{E}_\varphi (\neg q) \cup (\neg p \land \neg q) \\
&\quad \quad \quad \quad \text{then } \text{label}(s) := \text{label}(s) \cup \{A_\varphi[p \cup q]\}; \\
&\quad \text{end of cases;} \\
&\text{end of procedure;} \\
&\end{align*}
\]

---

**Figure 1:** Reduction Algorithm
Recursive Boolean Procedure AFC(C, \(\phi_0\))

\[
\phi = \bigwedge_{i=1}^{\infty} (\bigvee_{j=1}^{i} \phi_j)
\]

is a fairness constraint

output: true - if C is a fair component
false - otherwise **/

begin
1 if k=0 then return(true);
2 for i:=1 to k do
3 begin
4 p_occurs[i] := false;
5 for each s \(\in\) S do if C, s \(\models\) \(\phi_0\) \(p_i\) then p_occurs[i] := true;
6 if p_occurs[i]=false then
7 begin
8 \(\phi_0' := \bigwedge_{j=1}^{i-1} (\bigvee_{j=1}^{i} \phi_j) \bigwedge_{j=i+1}^{\infty} (\bigvee_{j=i+1}^{\infty} \phi_j)\);
9 C' := maximal substructure of C s.t. every states of C' satisfies \(q_i\);
10 X := \{D: D is a maximal strongly connected component of C'\};
11 for each D \(\in\) X do if FC(D, \(\phi_0'\))=true then return(true);
12 returns(false)
end
9 return(true)
end;

--------------------------------------------------------------------------

Figure 2: Fair Component Detection Algorithm
procedure AFS(M, \(\Phi_0\), S');
/* input: M=(S, R, L) is a structure, and
\[ \Phi_n = \bigcup_{i=1}^{\infty}(\Phi_1 \cup \cdots \cup \Phi_i) \]
output: S' -- the set of fair states of structure M */
begin
S' := \emptyset;
let X={C: C is a maximal strongly connected component of M};
for each C \in X do if AFC(C, \(\Phi_0\)) then S' := S' \cup \{s: s is a state of C\};

/* calculate the set of states in S which can reach some state in S' */
CLOSE := S';
while CLOSE \neq \emptyset do
begin
choose an arbitrary element t from CLOSE;
D := \{s: (s, t) \in R \land s \notin S'\};
S' := S' \cup D;
CLOSE := CLOSE \cup D \setminus \{t\}
end;
end;

Figure 3: Algorithm for Calculating Fair States.
Figure 4  Structure $M=(S, R, L)$
(a) A structure with arc labels

(b) A structure without arc labels

Figure 5
(a) A structure $M = (S, A_1, A_2, L)$

(b) $M' = (S', A', L')$

Figure 6
\[ x := 0; \quad b := \text{true}; \]

\[
\begin{array}{c}
\text{P}_1: \quad b \rightarrow x := x + 1; \\
\square \text{P}_2: \quad b \rightarrow b := \text{false}; \\
\end{array}
\]

(a)

(b)

(c)

(d)

Figure 7
(a) Global state transition graph for two processes mutual exclusion problem.

(b) Global state graph after termination of model checking algorithm

Figure 8