

Problem

Consider the differential equations

$$\frac{dx_1}{dt} = -x_2 + a_0 x_1^3 + a_1 x_1^2 x_2 + a_2 x_1 x_2^2 + a_3 x_2^3$$

$$\frac{dx_2}{dt} = x_1 + b_0 x_2^3 + b_1 x_2^2 x_1 + b_2 x_2 x_1^2 + b_3 x_1^3$$

Show that the solution $x_1 = x_2 = 0$ is asymptotically stable (i.e., there exists a $\delta > 0$ such that for every solution $\underline{x} = \underline{x}(t)$ with $\|\underline{x}(0)\| < \delta$ we have $\lim_{t \rightarrow \infty} \underline{x}(t) = \underline{0}$)

if $3a_0 + 3b_0 + a_2 + b_2 < 0$

and completely unstable (i.e., there exists a $\delta > 0$ such that for every solution $\underline{x} = \underline{x}(t)$ with $\underline{x}(0) \neq \underline{0}$ there exists a $T > 0$ such that $\|\underline{x}(t)\| > \delta$ for $t > T$)

if $3a_0 + 3b_0 + a_2 + b_2 > 0$.

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Solution

Consider the function

$$V(\underline{x}) = x_1^2 + x_2^2 + \frac{1}{2}(a_1 + b_3)x_1^4 - \frac{1}{2}(a_3 + b_1)x_2^4 - (a_0 - b_0)(x_1^3 x_2 + x_1 x_2^3) + \frac{1}{3}(a_2 + b_2)(x_1^3 x_2 - x_1 x_2^3).$$

Then along a trajectory $\underline{x} = \underline{x}(t)$ we have

$$\dot{V}(t) = \frac{d}{dt} V(\underline{x}(t)) = \frac{\partial V}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial V}{\partial x_2} \frac{dx_2}{dt} = [a_0 + b_0 + \frac{1}{3}(a_2 + b_2)](x_1^4 + x_2^4) + \text{terms of degree } 6.$$

It is obvious that V is a Lyapunov-function, i.e., V is continuously differentiable, $V(\underline{0}) = 0$, $V(\underline{x}) > 0$ in an open neighbourhood $\Omega(0 < \|\underline{x}\| < \delta)$ of $\underline{x} = \underline{0}$ and $\dot{V}(t) \leq 0$ for $\underline{x}(t) \in \Omega$ if $a_0 + b_0 + \frac{1}{3}(a_2 + b_2) \leq 0$.

The statements concerning stability and instability now follow from well-known theorems of Lyapunov (cf., e.g., La Salle and Lefschets, Stability by Liapunov's Direct Method, N.Y., 1961).

Note. The basic idea of the proof of Lyapunov's theorem is the following.

Let $0 < \delta_1 < \delta_2$, with sufficiently small δ_2 . Then the inequalities

$$\delta_1 \leq V(\underline{x}) \leq \delta_2$$

define a closed annular subdomain Ω_1 of Ω which contains the origin inside its inner boundary. If $\dot{V} < 0$ for $\underline{x}(t) \in \Omega$ then $\dot{V} \leq -c < 0$ as long as $\underline{x}(t) \in \Omega_1$. Hence if $\underline{x}(0) \in \Omega_1$, then certainly $V(\underline{x}(t)) < \delta_1$ for $t > (\delta_2 - \delta_1)/c$. Hence $\lim_{t \rightarrow \infty} V(\underline{x}(t)) = 0$, which implies $\lim_{t \rightarrow \infty} \underline{x}(t) = \underline{0}$. And similarly in the case of instability.