The problem of the maximum length of an ascending subsequence.

We consider a sequence of \( N \) elements \( A(1) \) through \( A(N) \). The order of increasing subscript value will be denoted by "the order from left to right". From such a sequence we can take so-called "subsequences of length \( s \)" by the removal of an arbitrary collection of \( N-s \) elements and retaining the remaining \( s \) elements in the order in which they occurred in the original sequence. When, in addition, each element has an integer value, we call a subsequence "ascending" iff it contains no element with a right-hand neighbour smaller than itself.

Note. According to this definition, all \( N \) subsequences of length 1 and even the empty subsequence are ascending ones. (End of note.)

We look for an algorithm that determines for any given sequence (with \( N > 0 \)) the maximum length of an ascending subsequence that can be taken from it.

Note. Although there need not be a unique longest ascending subsequence, the maximum length is unique, e.g. \( 3 \ 1 \ 1 \ 2 \ 4 \ 3 \) gives 4 for the maximum length, realized either by \( 1 \ 1 \ 2 \ 4 \) or by \( 1 \ 1 \ 2 \ 3 \). (End of note.)

If \( k \) represents the value we are looking for, we seek to establish the relation

\[
R: \quad k = \text{the maximum length of an ascending subsequence taken from } A(1) \text{ through } A(N).
\]

Because \( R \) contains the parameter \( N \), it is strongly suggested to take as invariant relation --or, as we shall see in a moment: as part of the invariant relation--

\[
P1(k, n): \quad k = \text{the maximum length of an ascending subsequence taken from } A(1) \text{ through } A(n).
\]

It has the virtues that it would do the job in the sense that

\[
(P1(k, n) \text{ and } n = N) \Rightarrow R \text{ and is easily established, e.g. by } k, n := 1, 1.
\]

These observations suggest to establish \( P1(k, n) \) for \( n = 1 \) and then to increase \( n \) under invariance of \( P1(k, n) \) until \( n = N \), more precisely: to increase \( n \) repeatedly by 1 and to restore each time, when destroyed, the truth of \( P1(k, n) \) by adjusting the value of \( k \). Because extension with a next element can never decrease the maximum length of an ascending subsequence and can increase it by at most 1, the adjustment of \( k \), when
needed, will have the form $k := k + 1$. More precisely: because

$$P1(k, n) = \text{wp}"n := n + 1", P1(k, n - 1)$$

we have to investigate after "$n := n + 1$" under which circumstances no adjustment of $k$ is needed, i.e. when

$$P1(k, n - 1) \Rightarrow P1(k, n)$$

and under what circumstances adjustment of $k$ is needed, i.e. when

$$P1(k, n - 1) \Rightarrow P1(k + 1, n)$$

Relation (2) holds iff $A(n)$ can be used to extend an ascending subsequence of maximum length ( = $k$) taken from $A(1)$ through $A(n - 1)$; this is true iff

$$A(n) \geq \text{the smallest rightmost value of an ascending subsequence of length } k \text{ taken from } A(1) \text{ through } A(n - 1).$$

This last inequality shows us, that besides $k$ -- as defined by $P1(k, n)$ -- we would also like to store the minimum rightmost value -- let us call it $m$ -- of an ascending subsequence of maximal length. If (2) holds, $k$ is obviously to be adjusted by $k := k + 1$, and the assignment $m := A(n)$ would make $m$ again equal to the minimum rightmost value of an ascending subsequence of maximal length (because, in this case, all ascending subsequences of maximal length taken from $A(1)$ through $A(n)$ will have $A(n)$ as their rightmost element.)

The introduction of $m$ as the minimum value of the rightmost value of an ascending subsequence of length $k$, presents, however, a problem in case (1). In that case, the extension with $A(n)$, although not leading to an increase of $k$, may require adjustment of $m$ as it may lead to a decrease of the minimum rightmost value of an ascending subsequence of that unchanged maximal length. This would be the case if the value $A(n)$ -- which now satisfies $A(n) \leq m$ -- could be used to extend an ascending subsequence of length $k - 1$, taken from $A(1)$ through $A(n - 1)$. In order to decide that, we would also need the minimum rightmost value of an ascending subsequence of length $k - 1$. Repeating the argument, we conclude that, instead of a scalar $m$, we need in addition to $k$ an array variable $m$ satisfying

$$P2(k, n, m): \text{ for all } j \text{ satisfying } 1 \leq j \leq k$$

$$m(j) = \text{the minimum rightmost value of an ascending subsequence of length } j \text{ and taken from } A(1) \text{ through } A(n).$$

Our total invariant relation will be $P1$ and $P2$. 
Again, for $n = 1$, $P2$ is easily initialized—with $m(1) = A(1)$--; we have to investigate, however, what updating obligations for the array variable $m$ are implied by our duty to keep $P2$ invariant. The crucial discovery in the analysis of our updating obligations for the array variable $m$ is that the elements of $m$ itself are ascending, more precisely:

$$1 \leq i < j \leq k \Rightarrow (m(i) \leq m(j))$$

This follows from the fact that $1 \leq i < j \leq k$ and $m(i) > m(j)$ leads to a contradiction: by removing from an ascending sequence of length $j$ and with $m(j)$ as its rightmost value the leftmost $j-i$ elements, an ascending sequence of length $i$ with $m(j)$ as rightmost value remains, and $m(i) > m(j)$ then contradicts $P2$.

Again we investigate the situation as reached after $n := n + 1$, i.e., when $P1(k, n - 1)$ and $P2(k, n - 1, m)$ holds. Relation (2) holds iff $A(n) \geq m(1)$. The new element $A(n)$ can be used to form a longer ascending sequence, $k$ has to be increased and the sequence of values is extended with $A(n)$ by

$$m \text{::next}(A(n))$$

it is correct to leave the values $m(i)$ with $1 \leq i < k$ unchanged, for the new element $A(n) \geq m(k)$ and can never give rise to a smaller rightmost value for any of the ascending sequences shorter than the new maximum length $k$.

Relation (1) holds iff $A(n) < m(k)$. Remembering that after the increase $n := n + 1$ the relation $P2(k, n - 1, m)$ holds, we have to answer the question: for which value(s) of $j$ is the minimum rightmost value of an ascending sequence of length $j$ take from $A(1)$ through $A(n)$ smaller than taken from $A(1)$ through $A(n-1)$? This can only be the case if $A(n)$ is its new rightmost element, which must be smaller than its old value $m(j)$. So we have

$$A(n) < m(j)$$

But $A(n)$ can only be the rightmost element of an ascending sequence of length $j$ if

$$j = 1 \text{ or } j > 1 \text{ and } m(j-1) \leq A(n)$$

(4)

Combining (3) and (4) we find

$$j = 1 \text{ iff } A(n) < m(1) \text{ and otherwise } j = \text{ the only(!) solution of } m(j-1) \leq A(n) < m(j)$$

This last solution is found in the program with a binary search; the invariant relation for the inner loop is $m(i) \leq A(n) < m(j)$.
Observing that $k = \text{m.hib}$, the current higher bound for the index, we can use for $m(k) = m(\text{m.hib})$ the usual abbreviation $\text{m.high}$ and conclude that we don't need the variable $k$ after all. For reasons of symmetry we denote $m(1)$ by $\text{m.low}$, as $\text{m.low} = 1$. Omitting all declarations we get the following program.

```plaintext
n := 1; m := (1, A(1));
do n \neq N ->
    n := n + 1;
    if A(n) \geq m.high ->
    m := \text{hiext}(A(n))
    [] A(n) < m.high ->
        if m.low > A(n) ->
            j := 1
            [] m.low \leq A(n) ->
                i, j := m.low, m.hib;
                do i \neq j - 1 ->
                    h := (i + j) \text{div} 2;
                    if m(h) \leq A(n) -> i := h
                    [] A(n) < m(h) -> j := h
                fi
            od
        fi;
    m := A(n)
    fi
od;
print(m.hib)
```

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