An experiment in mathematical exposition.

Many people feel attracted to the implication on account of the simplicity of the associated inference rules

\[
\begin{align*}
A &\Rightarrow B \\
B &\Rightarrow C \\
\hline
A &\Rightarrow C
\end{align*}
\]  

(1)

\[
\begin{align*}
A &\Rightarrow B \\
C &\Rightarrow D \\
\hline
A \land C &\Rightarrow B \land D
\end{align*}
\]

(2)

\[
\begin{align*}
A &\Rightarrow B \\
C &\Rightarrow D \\
\hline
A \lor C &\Rightarrow B \lor D
\end{align*}
\]

(3)

The transitivity of (1), and the symmetry of (2) and of (3) are clearly appealing. Rule (1), however, is a direct consequence of, and rules (2) and (3) are merely two different transcriptions of the same

\[
\begin{align*}
A \lor B \\
C \lor D \\
\hline
A \lor C \lor (B \land D)
\end{align*}
\]

(4)

a rule, which - on account of the symmetry of the disjunction - can be applied in four different ways to the two given antecedents. Rules (2) and (3) give only two of the four. Rule (1) emerges as the special case
\[ A \lor B \\
C \lor \neg B \\
\hline
A \lor C \] 

I called this "a direct consequence" because perhaps somewhat arbitrarily - I would like to distinguish between inference rules (different applications of which may yield results that are not equivalent) and simplifications that are possible according to boolean algebra - such as replacing \(B \land \neg B\) by false and \(A \lor C \lor \text{false}\) by \(A \lor C\) - but never change the value of the boolean expression.

* * *

The above caused me to revisit the problem of the nine mathematicians visiting an international congress, and about whom we are invited to prove

\[ A \lor B \lor C \] (5)

with

A: there exists a triple of mathematicians that is incommunicado (i.e. such that no two of them have a language in common)

B: there exists a mathematician mastering more than three languages

C: there exists a language mastered by at least three mathematicians.

Very much like the introduction of (named!) auxiliary lines or points in geometry proofs, I propose to introduce named auxiliary propositions, such that
we can prove lemmata connecting them to the above propositions, such as

D: there exists a mathematician that can communicate with more others than he masters languages,

for which we can prove

**Lemma 1.** $C \lor D$.

**Proof.** Obvious. With this qualification we mean here that we can start as well with observing $C \lor "each mathematician communicates in different languages with those others he can communicate with", etc.

as with observing $D \lor "there exists a mathematician that shares a language with at least two others", etc.

(End of proof of Lemma 1.)

With

E: there exists a mathematician that can communicate with more than three others,

we can prove

**Lemma 2.** $A \lor E$.

**Proof.** Let "$x|y$" here stand for "$x$ and $y$ are two different mathematicians that have no language in common. With

G: for each $x$, the equation $x|u$ has at least five different solutions for $u$,

we observe (obviously) $E \lor G$.  

(6)

With
with \( y \) and \( z \) constrained to belong to an
arbitrary quinuple, the equation \( y/z \) has
at least one solution in \( y \) and \( z \),
we observe (equally obviously)
\[
E \lor H
\]
Applying rule (4) to assertions (6) and (7) we
find
\[
E \lor (G \lor H)
\]
hence
\[
E \lor "\text{for each } x, \text{the equation } x/y \land x/z \land y/z
\text{has at least one solution in } y \text{ and } z".
\]
(End of proof of Lemma 2)

Applying rule (4) to Lemmata 1 and 2 we
infer the
Corollary
\[
A \lor C \lor (E \lor D)
\]
Remembering rule (4) we see that (5) has
been proved when we can prove \( B \lor (E \lor D) \)
or, equivalently
Lemma 3: \( B \lor D \lor E \).
Proof: Obvious. (End of proof of Lemma 3).

* * *
Note that in the above the Corollary was only
used for heuristic purposes. Once Lemmata
1, 2, and 3 have been established we could have
inferred
\[
A \lor E \quad \text{and} \quad A \lor B \lor D
\]
\[
B \lor D \lor E \quad \Longrightarrow \quad C \lor D
\]
\[
A \lor B \lor D \quad \Longrightarrow \quad A \lor B \lor C
\]
and our two individual inferences would have been of the traditional form of the transitive implication.

* * *

I know that firm believers in the so-called "natural deduction" will state that, in the case of Lemma 2, I am just "deducing naturally" that A follows from the "assumption" 1E. In this appreciation they will find themselves strengthened by the observation that in that proof all assertions start with "Ev". They have a point, but the point is weak. Look at the structure of the proof as a whole. Lemmata 1, 2, and 3 capture it; from there rule (4) does the job, and at that level it is very arbitrary to subdivide assertions into assumptions and conclusions.

Remark. Observing the seven triples xyz for a pair (x,y), the argument proving Lemma 2 can equally well be phrased in terms of assertions starting with "A v". In the sense used above, also Lemma 2 is obvious. (End of remark.)

Plataanstraat 5
5671 AL NUENEN
The Netherlands

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prof. dr. Edsger W. Dijkstra
Burroughs Research Fellow