Two cheers for equivalence

Let us consider the operators $\neg$, $\lor$, and $\equiv$; $\neg$ is a unary operator, $\lor$ and $\equiv$ are symmetric and associative operators defined on bags of at least 2 operands. For the latter two we adopt the usual infix notation; the three operators have been listed in the order of decreasing syntactic binding power.

In the presence of $P \equiv R$ a new formula may be formed by replacing in an existing formula one or more occurrences of $P$ by $R$. (Leibniz's Rule.)

**Axiom 0**

$$P \equiv P \equiv Q \lor \neg Q$$

Parsing this ($P \equiv P \equiv Q \lor \neg Q$) we see that $Q \lor \neg Q$ may be replaced by $P \equiv P$, which does not depend on $Q$! This suggests to introduce the abbreviation

**Abbr. 0**

$$Q \lor \neg Q \equiv \text{black}$$

(where "black" may be viewed as a constant).

Applying Leibniz's Rule to the above two formulae we generate

**Theorem 0**

$$P \equiv P \equiv \text{black}$$

Parsing this as $P \equiv (P \equiv \text{black})$, and applying Leibniz's Rule we see that the suffix $\equiv \text{black}$ can be removed from a formula that ends on it; we are also free to add it to an existing formula. So we derive from Theorem 0 and Abbr.0 respectively

**Theorem 1**

$$P \equiv P$$

**Theorem 2**

$$Q \lor \neg Q$$

We now add

**Axiom 1**

$$P \lor \neg Q \equiv P \lor Q \equiv P$$

Substituting $P$ for $Q$ in Axiom 1 yields

$$\text{black} \equiv P \lor P \equiv P$$
yielding with Theorem 0

**Theorem 3** \( P \lor P \equiv P \). 

Applying Theorem 3 to Abbr. 0 yields 
\( Q \lor Q \lor \neg Q \equiv \text{black} \), 

yielding with Abbr. 0

**Theorem 4** \( P \lor \text{black} \equiv \text{black} \). 

Substituting black for \( Q \) in Axiom 1 yields 
\( P \lor \neg \text{black} \equiv P \lor \text{black} \equiv P \) 

and by application of Theorems 4 and 0

**Theorem 5** \( P \lor \neg \text{black} \equiv P \). 

Substitution of \( \neg \text{black} \) for \( P \) in Axiom 1 yields 
\( \neg \text{black} \lor \neg Q \equiv \neg \text{black} \lor Q \equiv \neg \text{black} \) 

and by applying Theorem 5 twice we get

**Theorem 6** \( \neg Q \equiv Q \equiv \neg \text{black} \). 

Substitution of \( \neg Q \) for \( Q \) yields 
\( \neg \neg Q \equiv \neg Q \equiv \neg \text{black} \) 

and from the latter two we get with Leibniz's Rule

**Theorem 7** \( \neg \neg Q \equiv Q \). 

Substituting in Axiom 1 \( P \lor Q \) for \( Q \), we get
\( P \lor \neg (P \lor Q) \equiv P \lor P \lor Q \equiv P \)

yielding with Theorem 3
\( P \lor \neg (P \lor Q) \equiv P \lor Q \equiv P \).

Confronting this with Axiom 1, we get

**Theorem 8** \( P \lor \neg (P \lor Q) \equiv P \lor \neg Q \).
Substituting \( P \equiv Q \) for \( Q \) in Theorem 6 we get
\[
\neg(P \equiv Q) \equiv P \equiv Q \equiv \neg \text{black}
\]
and applying Theorem 6 once more we generate

**Theorem 9** \( \neg(P \equiv Q) \equiv P \equiv \neg Q \).

With Theorems 1 and 6 we generate in succession
\[
\neg \text{black} \equiv \neg \text{black} \equiv \text{black}
\]
\[
P \equiv \neg P \equiv R \equiv \neg R \equiv Q \equiv \neg Q \quad \text{i.e.}
\]

**Theorem 10** \( P \equiv Q \equiv R \equiv \neg P \equiv Q \equiv \neg R \).

Substitution of \( \neg P \) for \( P \) in Axiom 1 yields
\[
\neg P \lor \neg Q \equiv \neg P \lor Q \equiv \neg P
\]
With Theorem 10 this yields
\[
\neg(\neg P \lor \neg Q) \equiv \neg P \lor Q \equiv P
\]
and with

**Abbr. 1** \( \neg(\neg P \lor \neg Q) \equiv P \land Q \).

**Theorem 11** \( P \land Q \equiv \neg P \lor Q \equiv P \).

In the sequel, appeals to Abbr. 1 and Theorem 7 will often be summarized by referring to the Law of de Morgan.

Substitution of \( P \land Q \) for \( Q \) in Axiom 1 yields
\[
P \lor (P \land Q) \equiv P \lor (P \land Q) \equiv P
\]
With de Morgan's Law
\[
P \lor \neg P \lor \neg Q \equiv P \lor (P \land Q) \equiv P
\]
With Abbr. 0, Theorems 4 and 0 we generate

**Theorem 12** \( P \lor (P \land Q) \equiv P \).
Interchanging in Axiom 1 \( P \) and \( Q \) gives
\[ Q \lor P \equiv P \lor Q \equiv Q \]
which yields with Axiom 1

\textbf{Theorem 13} \( P \equiv Q \equiv Q \lor \neg P \equiv P \lor \neg Q \)

Applying Theorem 12 we derive from Theorem 13
\[ P \equiv Q \equiv Q \lor (Q \land P) \lor \neg P \equiv P \lor \neg Q \]
with de Morgan's Law
\[ P \equiv Q \equiv \neg (P \lor Q) \lor (Q \lor P) \equiv P \lor \neg Q \]
and applying Theorem 11, we generate

\textbf{Theorem 14} \( P \equiv Q \equiv (P \lor \neg Q) \land (Q \lor \neg P) \)

Note Theorem 14 corresponds to the Hilbert-Ackermann definition of equivalence. (End of Note.)

From Theorem 4 we derive
\[ Q \lor \neg R \lor \text{black} \]
from which we generate with Abbr. 0
\[ Q \lor P \lor R \lor \neg P \]
which yields with Theorem 8 (twice)
\[ Q \lor (Q \lor P) \lor R \lor (R \lor P) \]
which yields with de Morgan's Law

\textbf{Theorem 15} \( (Q \land P) \lor (R \land \neg P) \lor (Q \lor R) \)

Note Theorem 15 corresponds to the last axiom of Hilbert-Ackermann

\[ (P \Rightarrow Q) \Rightarrow ((P \lor R) \Rightarrow (Q \lor R)) \]

(End of Note.)
Now comes a trivial section that I shall only indicate. With

\[ \neg \text{black} \equiv \text{white} \]

we leave it to the reader to generate—mostly with

de Morgan’s law—all sorts of useful theorems such as

\[ \neg Q \land \neg Q \equiv \text{white} \]
\[ P \land P \equiv P \]
\[ P \land \text{white} \equiv \text{white} \]
\[ P \land \text{black} \equiv \neg P \]
\[ P \land (\neg P \lor Q) \equiv P \land Q \]
\[ P \land (P \lor Q) \equiv P \]

So far I did not succeed in generating, say

\[ (P \equiv Q) \land (P \equiv R) \equiv (P \equiv Q) \land (Q \equiv R) \]

or the distributivity of \( \land \) and \( \lor \). I have tried whether I could modify my axioms—currently, none of them contains three variables—but did not succeed. The obvious alternative is the generalization of Leibniz’s Rule: if \( Q \) could be generated in the additional presence of \( P \), we allow ourselves to generate \( \neg P \lor Q \).

Since I don’t want to become a logician I had better stop; in any case I have had my fun.

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