A universal quantification revisited

by C.S. Scholten and Edsger W. Dijkstra

For any bag B and any boolean function \( b \) defined on the elements of B we deem \( (\forall X: X \in B: bX) \) defined as usual. Our first purpose is to define as a predicate

\[
(\forall X: X \in B: fX)
\]

where \( f \) is a function from the elements of B to predicates on some space. (If the elements of B are predicates on that same space, \( f \) is traditionally known as a predicate transformer.)

A traditional way of defining the predicate \( (\forall) \) is by point-wise definition: in each point of state space the \( fX \) stand for boolean values for which universal quantification over B is defined. We should like to develop a predicate calculus as far as possible without explicit reference to points of the space; we would like to define it by means of a set of rules for the manipulation of formulae.

We postulate expressions of form \( (\forall) \) to satisfy the following two rules—where a pair of square brackets, if so desired, may be interpreted as universal quantification over the points of space—

\[
(\forall X: X \in B: [fX]) \equiv [((\forall X: X \in B: fX)]
\]

\[
[([\forall X: X \in B: Q \lor fX]) \equiv Q \lor [((\forall X: X \in B: fX)]
\]

for
any predicate \( Q \).

Note that in the case of a finite bag \( B \), (1) and (2) are consistent with the interpretation of (0) as the conjunction of \( f_X \) over the elements of \( B \).

For the sake of brevity, the range "\( X \in B \)" will be omitted in the sequel.

We observe for any \( Z \)
\[
\text{true} = \{ \text{substitution of } Z \text{ for } Q \text{ in (2)} \} \\
[(AX::\, \neg Z \lor f_X) \equiv \neg Z \lor (AX::\, f_X)] \\
\Rightarrow \{ \text{Leibniz's Rule} \} \\
[(AX::\, \neg Z \lor f_X)] \equiv [(\neg Z \lor (AX::\, f_X)] \\
= \{(1) \text{ and the definition of } \Rightarrow \} \\
(3) \quad (AX::\, [Z \Rightarrow f_X]) \equiv [Z \Rightarrow (AX::\, f_X)]
\]

One of the consequences of (3) is that (0) is the weakest solution of
\[
(4) \quad Z::(AX::\, [Z \Rightarrow f_X])
\]
because, firstly, any solution of (4) substituted for \( Z \) in (3) reduces its left-hand side to true and, hence, implies (6), and, secondly, (0) substituted for \( Z \) in (3) reduces its right-hand side to true and, hence, is a solution of (4).

Consider in addition to \( B \) a bag of elements \( Y \) and let \( g \) be a predicate-valued on the Cartesian product of \( B \) and \( C \). For any \( Z \)
true
= \{ \text{substitution of } g \ XY \text{ for } P \ X \text{ in (3)} \}
(AY::(AX::[Z \Rightarrow g \ XY])) \equiv [Z \Rightarrow (AX::g \ XY)])
\Rightarrow \{ \text{predicate calculus} \}
(AY::(AX::[Z \Rightarrow g \ XY])) \equiv (AY::[Z \Rightarrow (AX::g \ XY)])
= \{ \text{predicate calculus} \}
(AX::(AY::[Z \Rightarrow g \ XY])) \equiv (AY::[Z \Rightarrow (AX::g \ XY)])
= \{ (3) \}
(AX::[Z \Rightarrow (AX::g \ XY)]) \equiv (AX::[Z \Rightarrow (AX::g \ XY)])
= \{ (3) \text{ applied to both sides} \}
[Z \Rightarrow (AX::(AY::g \ XY))] \equiv [Z \Rightarrow (AY::(AX::g \ XY))]

Hence
(5) \quad [ (AX::(AY::g \ XY)) \equiv (AY::(AX::g \ XY))] ,
i.e. also when the terms are predicates, the order of universal quantifications is immaterial.

\textit{Corollary of (5)}
\[(AX::P \ X \wedge g \ X) \equiv (AX::P \ X) \wedge (AX::g \ X)] .

Note that from this corollary the monotonicity of universal quantification over \(X\) follows.

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Slightly shifting notational gears we consider (6) with \(f\) the identity function and for \(B\) the bag of solutions of the equation

(6) \quad X: [g \ X] ,

where \(g\) is some predicate transformer, i.e. \(X\) and \(gX\) are predicates on the same space. For the sake
of reference, the resulting expression is denoted by \( Q \), i.e.

\[
(7) \quad [Q \equiv (AX: [gX]: X)]
\]

Rewriting (3) we observe for any \( Z \)

\[
(8) \quad (AX: [gX]: [Z \Rightarrow X]) \equiv [Z \Rightarrow Q]
\]

with the corollary

\[
(9) \quad (AX: [gX]: [Q \Rightarrow X])
\]

We are now ready to prove

**Lemma 0.** In terms of (6) and (7) the following three assertions are equivalent:

(i) \( Q \) is a solution of (6)
(ii) \( Q \) is the strongest solution of (6)
(iii) a strongest solution of (6) exists.

**Proof.** Formally expressed the assertions are:

(i) \( [gQ] \)

(ii) \( [gQ] \land (AX: [gX]: [Q \Rightarrow X]) \)

(iii) \( (EP: [gP]: (AX: [gX]: [P \Rightarrow X])) \)

The equivalence \((i) \equiv (iii)\) follows immediately from (9). Further we observe

\[
(iii)
\]

\[
= \{ \text{definition of (iii) and (8) with } P \text{ for } Z \}
\]

\[
(EP: [gP]: [P \Rightarrow Q])
\]

\[
= \{ \text{on account of (9)} \}
\]

\[
(EP: [gP]: [P \Rightarrow Q] \land [Q \Rightarrow P])
\]
\[
\left\{ \text{predicate calculus} \right\} \\
\{ \text{EP: } [gP]: [P \equiv Q] \} \\
\left\{ \text{predicate calculus} \right\} \\
[gQ] \\
= \left\{ \text{definition of } (c) \right\} \\
(c) \\
(\text{End of Proof.})
\]

Remark 0. Note that Lemma 0 holds without any further assumptions about predicate transformer \(g\).
(End of Remark 0.)

We mention a special consequence of Lemma 0. In order to prove that (6) has a strongest solution, one shows for instance that \(Q\) — the conjunction of all solutions of (6) — is a solution of (6). Sometimes one can prove a somewhat stronger property of \(g\), viz. that the conjunction of any bag of solutions of (6) is again a solution of (6). Such is the case in the following two examples.

**Example 0.** For monotonic \(h\), the equation

\[(10) \quad X : [hX \Rightarrow X]\]

has a strongest solution.

**Proof.** In view of the above it suffices to show that \((AX : X \in B : X)\) solves (10) for \(B\) any bag of solutions of (10).

\[
= \left\{ \text{definition of } B^3 \right\} \\
(AX : X \in B : [hX \Rightarrow X])
\]
\[ \Rightarrow \{ \text{monotonicity of universal quantification} \} \\
\quad [\langle AX: X \in B: hX \rangle \Rightarrow \langle AX: X \in B: X \rangle] \\
\Rightarrow \{ \text{monotonicity of } h, \text{ see next page} \} \\
\quad [h(AX: X \in B: X) \Rightarrow (AX: X \in B: X)] \\
\hspace{1cm} (\text{End of Proof.}) \\
\] (Note that Example 0 states "half" of the Theorem of Knaster-Tarski.)

**Example 1.** For universally conjunctive \( h \), the equation

\[(11) \quad X: [P \lor hX] \]

has a strongest solution for any predicate \( P \).

**Proof.** For any bag \( B \) of solutions of (11) we have

\[
\text{true} \\
= \{ \text{definition of } B \} \\
\quad (AX: X \in B: [P \lor hX]) \\
= \{ \text{predicate calculus} \} \\
\quad [AX: X \in B: P \lor hX] \\
= \{ \text{on account of (2)} \} \\
\quad [P \lor (AX: X \in B: hX)] \\
= \{ \text{universal conjunctivity of } h \} \\
\quad [P \lor h(AX: X \in B: X)] \\
\hspace{1cm} (\text{End of Proof.})
\]
The last transition in the proof of Example 0 relies on

(11) For monotonic $h$ and any bag $B$
\[ [h \, (\exists X : X \in B : X) \Rightarrow (\exists X : X \in B : hX)] \]

Proof
true
\[ = \{ \text{with } f \text{ the identity and } (\forall Y : Y \in B : Y) \text{ for } Z \} \]
\[ (\exists X : X \in B : [h \, (\exists Y : Y \in B : Y) \Rightarrow X]) \]
\[ \Rightarrow \{ \text{monotonicity of } h \} \]
\[ (\exists X : X \in B : [h \, (\exists Y : Y \in B : Y) \Rightarrow hX]) \]
\[ = \{ (3) \} \]
\[ [h \, (\exists Y : Y \in B : Y) \Rightarrow (\exists X : X \in B : hX)] \]
\[ = \{ \text{renaming the dummy} \} \]
\[ (11) \]

(End of Proof.)

* * *

The conclusion we drew from the Corollary of (5), viz. the monotonicity of universal quantification, was a bit rash. The statement of the monotonicity—and in this version we used it in the proof of Example 0—is

(12) \[ (\exists X : X \in B : [fX \Rightarrow gX]) \Rightarrow [ (\exists X : X \in B : fX) \Rightarrow (\exists X : X \in B : gX) ] \]

For the formal proof of (12) we need an extension of Leibniz's Rule, viz.

(13) \[ (\exists X : X \in B : [fX \equiv gX]) \Rightarrow [ (\exists X : X \in B : fX) \equiv (\exists X : X \in B : gX) ] \]

The proof of (12) is then as follows—for the sake of brevity under omission of the range—
Proof

\[(\forall X:: [fX \equiv gX])\]
\[\Rightarrow \{\text{predicate calculus}\}\]
\[\{(\forall X:: [fX \equiv fX \land gX])\}\]
\[\Rightarrow \{(13)\}\]
\[\{(\forall X:: fX) \equiv (\forall X:: fX \land gX)\}\]
\[\Rightarrow \{\text{Corollary of (5)}\}\]
\[\{(\forall X:: fX) \equiv (\forall X:: fX) \land (\forall X:: gX)\}\]
\[\Rightarrow \{\text{predicate calculus}\}\]
\[\{(\forall X:: fX) \Rightarrow (\forall X:: gX)\}\]

(End of Proof.)

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drs. C.S. Scholten
Scientific Advisor
Philips Research Laboratories
5600 JA EINDHOVEN
The Netherlands

prof. dr. Edsger W. Dijkstra
Burroughs Research Fellow
Plataanstraat 6
5671 AL NUENEN
The Netherlands