The maximum length of a segment satisfying a monotonic predicate

For a given sequence \( f(i:0 \leq i \leq N) \), \( f(i:x \leq i:y) \) with \( 0 \leq x \leq y \leq N \) is called "a segment of length \( y-x \)."

Let, with \( 0 \leq x \leq y \leq N \), \( B \times y \) be some predicate on segment \( f(i:x \leq i:y) \) such that

\[
A(x,h,k,y:0 \leq x \leq h \leq k \leq y \leq N: B \times k \lor \neg B \times y)
\]

such a predicate is called "monotonic". We know many examples of monotonic predicates, such as:

- all elements positive
- all elements equal
- ascending
- not containing adjacent, non-empty, equal subsegments
- having its first differences of alternating signs.

For a \( B \) that holds for any empty segment we shall derive a program establishing \( R \) given by

\[
R: \quad c = \text{MAX}(x,y:0 \leq x \leq y \leq N \land B \times y: y-x)
\]

To begin with we approach the problem in the standard way by introducing a variable \( n \) satisfying \( P_0 \) given by

\[
P_0: \quad c = \text{MAX}(x,y:0 \leq x \leq y \leq n \land B \times y: y-x) \land 0 \leq n \leq N,
\]

which yields a program of the structure
\[ \text{\textcopyright{} n: int ; c, n := 0, 0 \{ invariant Po \} \} } \\
\begin{align*}
\text{do } & n \neq N \rightarrow \\
& \quad \text{"increase n by 1 under invariance of Po"} \\
& \end{align*}
\] \\
\].

From \( P_0 \land n \neq N \) we conclude
\[
\begin{align*}
\max_{(n+1 - \min(x: 0 \leq x \leq n+1, + B \times (n+1): n+1 - x)} \max c \ ,
\end{align*}
\]
for the sake of convenience we rewrite the last line as
\[
\begin{align*}
(n+1 - \min(x: 0 \leq x \leq n+1, + B \times (n+1): x)) \max c 
\end{align*}
\]
which suggests the introduction of a variable \( h \) satisfying \( P_i \), given by
\[
\begin{align*}
P_i: \quad h &= \min(x: 0 \leq x \leq n \land B \times n: x) \ .
\end{align*}
\]

This yields a program of the structure
\[
\begin{align*}
\text{\textcopyright{} n, h: int ; c, n, h := 0, 0, 0 \{ invariant P_0 \land P_i \} \} } \\
\begin{align*}
\text{do } & n \neq N \rightarrow \\
& \quad \text{"establish } P_i(n+1/n)" \\
& \quad c := (n+1 - h) \max c \ \{ P_0(n+1/n) \} \\
& \quad n := n + 1 \ \{ P_0 \land P_i \}
\end{align*}
\] \\
\].

Without exploiting any property of \( B \) (beyond the fact that it holds for the empty segment), the Linear Search Theorem tells us that there is only one way of establishing \( P_i(n+1/n) \), viz.
\[ h := 0 ; \text{do} \cap B \ h (n+1) \rightarrow h := h + 1 \ \text{od} \]

which disregarding the evaluations of \( B \) gives in general rise to a quadratic algorithm.

From the monotonicity of \( B \), however, we can conclude that the solution of the equation \( h: P_1 \) is at most the solution of \( h: P_1(n+1/n) \); hence, establishing \( P_1(n+1/n) \) can be implemented by

\[ \text{do} \cap B \ h (n+1) \rightarrow h := h + 1 \ \text{od} \]

which again disregarding the evaluations of \( B \) gives rise to a linear algorithm.

\textbf{Note.} From the above analysis follows that the monotonicity requirement on \( B \) is stronger than necessary: a "one-sided" monotonicity

\[ A(x, k, y: 0 \leq x \leq k \leq y \leq N: B \times k \cap \neg B \times y) \]

would have sufficed. An example of such a \( B \) is

\[ B \times y \equiv A(j: x \leq j < y: f \times \leq f j) \]

(End of Note.)

\[ * \quad * \quad * \]

Three remarks are in order. We have postulated that \( B \) holds for the empty segment because — see \( R \) — we did not care to define \( \text{MAX} \) over an empty bag. If \( B \) holds for any one-element segment, it is often convenient to deal with \( N=0 \) separately; for \( N>0 \), the repetition can then be initialized with \( n=1 \) and has \( h<n \) as a further invariant.
Secondly, the analytical structure of $B$ is, thanks to some transitivity, often such that the net effect of
\[
\text{do } \{ \text{B } \text{h } (n+1) \rightarrow \text{h:=h+1} \} \quad \text{od}
\]
can be captured by a modest alternative statement, say of the form
\[
\text{if } \ldots \rightarrow \text{skip } \| \ldots \rightarrow \text{h:=n } \text{fi} \quad .
\]
Thirdly, the assignment statement
\[
c := (n+1 - h) \max c
\]
is equivalent to a skip in the case $n+1 - h \leq c$, a situation implied by $N - h \leq c$. Once established, however, $N - h \leq c$ is an invariant of the repetition; hence we can strengthen the guard by its negation $h+c < N$. But since $n \leq h+c$ is a further invariant of the repetition $n \neq N \land h+c < N$ can be simplified to just $h+c < N$.

* * *

By way of illustration we give the solution for
\[
B \times y \equiv A \langle j : x \leq j < y : f_x \leq f_j \rangle.
\]
\[
\text{if } N = 0 \rightarrow c := 0
\]
\[
\| N > 0 \rightarrow [\text{n, h : int;} c, n, h := 1, 1, 0
\quad \text{do } h+c < N \rightarrow
\quad \text{if } f(n) \geq f(h) \rightarrow \text{skip } \| f(n) < f(h) \rightarrow h:=n \text{ fi}
\quad ; \text{n:=n+1; c := (n-h) max c}
\quad \text{od}
\quad ]
\]
\[
\text{fi} \quad .
\]
The above B is one of one-sided monotonicity. Had we chosen

\[ B \times y \equiv A( j : x \leq j < y : f_x = f_j ) \]

we would have posed the Plateau Problem (see [0], p. 203, which deals with the special case that the given sequence is ordered). Its solution is obtained by replacing the inner alternative statement in the above by

\[ \text{if } f(n) = f(h) \rightarrow \text{skip } \text{ if } f(n) \neq f(h) \rightarrow h := n \]


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