\[ |x_n| = x_{n-1} + x_{n+1} \text{ has period } 8. \]

This morning I heard the at first sight surprising theorem that the sequence of real numbers \( x_n \) \((-\infty < n < \infty)\) such that \(|x_n| = x_{n-1} + x_{n+1}\) has a period of length 8. Here is my proof.

Since \( x_{n-1} + x_{n+1} \geq 0 \), there exists an element \( x_i \) such that \( x_i \geq 0 \). Since \( x_i + x_{i+1} \geq 0 \), \( x_{i-1} \geq 0 \) or \( x_{i+1} \), i.e. the sequence contains two successive elements \( p \) and \( q \), such that \( p \geq 0 \) and \( q \geq 0 \). Without loss of generality we may choose \( p < q \). But this means that the sequence contains 3 consecutive nonnegative elements \( p \), \( q \), \( q-p \), or, after renaming \( p \), \( p+r \) or \( p-r \) for nonnegative \( p \) and \( r \). Let \( r < p \) and let us extend the sequence in the direction of \( r \) - since the relation for \( x_n \) is symmetric in \( x_{n-1} \) and \( x_{n+1} \), the direction of indexing is irrelevant.

With \( x_0 = x_9 \) and \( x_1 = x_{10} \), the theorem has been proved without case analysis.

\[ \begin{align*}
x_0 &= p & (x_0 \geq 0) \\
x_1 &= p+r & (x_1 \geq 0) \\
x_2 &= r & (x_2 \geq 0) \\
x_3 &= -p & (x_3 \leq 0) \\
x_4 &= p-r & (x_4 \geq 0) \\
x_5 &= 2p-r & (x_5 \geq 0) \\
x_6 &= p & (x_6 \geq 0) \\
x_7 &= r-p & (x_7 \leq 0) \\
x_8 &= -r & (x_8 \leq 0) \\
x_9 &= p & (x_9 \geq 0) \\
x_{10} &= p+r & (x_{10} \geq 0)
\end{align*} \]

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