A sequence with \(|x_n| = x_{n-1} + x_{n+1}\) has period 9.

Recently I heard the theorem that any (in both directions) infinite sequence of real numbers \(x_n\) such that for all \(n\)

\[|x_n| = x_{n-1} + x_{n+1}\]  

(0)

has a period of length 9. Here is my proof.

From (0) we conclude (i) that the sequence contains a nonnegative element, (ii) that one of its neighbours is nonnegative, and (iii) that at least one of the two elements adjacent to a pair of nonnegative neighbours is nonnegative. More precisely:

- the sequence contains in some direction a triple of adjacent elements
- of the form \((p, p+r, r)\) with
- \(0 \leq r \leq p\). To the left we have extended
- the sequence with another 8 elements.

From (0) we further conclude that the whole sequence is determined by a pair of adjacent values; hence, the repetition of the pair \((p, p+r)\) at distance 9 proves the theorem. [The above deserves recording for its lack of case analyses.]

Plataanstraat 5
5671 AL NUENEN
The Netherlands

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prof. dr. Edsger W. Dijkstra
Burroughs Research Fellow.