Why the importance of continuity seems to be overrated

Without an appeal to continuity and without using
fixed-point induction, we shall prove the following
theorem.

**Theorem**  Let \((C, <)\) be a well-founded set. Let
predicates \(B\) and \(P\), statement \(S\), and function
\(t\) be such that

\[
[P \Rightarrow t \in C]
\]

and with fresh "thought variable" \(y\)

\[
[B \land P \Rightarrow \text{wp}(\text{"y=t"}, \text{wp}(S, P \land t < y))] \tag{1}
\]

Then

\[
[P \Rightarrow \text{wp}(\text{"do B \rightarrow S od"}, \text{true})] \tag{2}
\]

in which the right-hand side is defined as the
strongest solution of

\[
X: [(B \land \text{wp}(S, X)) \lor \neg B \equiv X] \tag{3}
\]

**Proof.** Equation (3) has a strongest solution since
\(\text{wp}(S, \_?)\) is conjunctive and, hence, monotonic. Let
\(X\) be the strongest solution of (3). Since we can
conclude from (0)

\[
[P \Rightarrow (\exists x: x \in C: t \leq x)] \tag{2} - i.e. [P \Rightarrow X] - is proved by demonstrating
\[
[P \land (\exists x: x \in C: t \leq x) \Rightarrow X]
\]
or, equivalently,
(Ax: x in C: [P ∧ t ≤ x ⇒ X]) .  

In view of C's well-foundedness, (4) will be shown by mathematical induction, i.e. for an x in C, we shall derive [P ∧ t ≤ x ⇒ X] under the hypothesis [P ∧ t < x ⇒ X].

We observe for any Z

\[ Z \equiv B \land P \land t \leq x \]

\[ \Downarrow \{ (1) \}\]

\[ Z \equiv wp("y:=t", wp(S, P \land t < y)) \land t \leq x \]

= \{ Axiom of Assignment; conjunctivity of wp \}

\[ Z \equiv wp("y:=t", wp(S, P \land t < y) \land y < x)) \]

= \{ wp(S, Q) \land y < x \equiv wp(S, Q \land y < x)) for any Q since y and x are thought variables \}

\[ Z \equiv wp("y:=t", wp(S, P \land t < y \land y < x))) \]

\[ \Downarrow \{ \text{monotonicity of wp; transitivity of } < \}\]

\[ Z \equiv wp("y:=t", wp(S, P \land t < x)) \]

\[ \Downarrow \{ y \text{ is a thought variable} \}

\[ Z \equiv wp(S, P \land t < x) \]

\[ \Downarrow \{ \text{Hypothesis and monotonicity of wp} \]\n
\[ Z \equiv wp(S, X) \]

Eliminating Z, we conclude under the hypothesis

\[ [B \land P \land t \leq x \Rightarrow B \land wp(S, X)] \]

Furthermore we have

\[ \neg B \land P \land t \leq x \Rightarrow \neg B \]

Hence

\[ P \land t \leq x \Rightarrow (B \land wp(S, X)) \lor \neg B \]

and, since X is a solution of (3),
\[ [P \land t \leq x \Rightarrow x] . \]

(End of Proof.)

The theorem is well-known for or-continuous \(wp(S,?)\) and natural \(t\). The continuity permits us to write the strongest solution of \((3)\) in closed form, viz. as the limit of a weakening chain. I \((\ast E W D)\) used this expression a decade ago to prove the restricted theorem, but that proof was by no means simpler than our current one.

The above proof casts serious doubts on the supposed need of fancy things such as transfinite induction for reasoning about programs with unbounded nondeterminacy (as we might, for instance, encounter in an abstract program containing the unrefined statement "establish \(P\)" or with fair interleaving of the atomic actions of concurrent programs). This is a very nice thought.

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