Well-founded sets revisited

In the following

\( (C, \prec) \) is a nonempty partially ordered set,
\( x, y \) are elements of \( C \),
\( S \) is a subset of \( C \),
\( P, Q \) are predicates on \( C \),
where \( S \) and \( P \) are coupled by
(0) \( \forall x \in S \), or \( S = \{ x \mid \neg P x \} \).

As a result we have
(1) \( S = \emptyset \equiv (\forall x : P x) \).

"\( x \) is a minimal element of \( S \)" means
(2) \( x \in S \wedge (\forall y : y < x : \neg y \in S) \).

"\( C \) is well-founded" means "any nonempty subset of \( C \) contains a minimal element".

**Theorem 0** "\( C \) is well-founded" \( \equiv \) "mathematical induction over \( C \) is valid".

**Proof**

"\( C \) is well-founded"
= \{ definitions of well-foundedness and minimal element \}
\( (\exists S : S \neq \emptyset \equiv (\forall x : x \in S \wedge (\exists y : y < x : \neg y \in S))) \)
= \{ \text{predicate calculus, de Morgan in particular}\}

\begin{align*}
\forall S :: S &= \emptyset \equiv \\
\forall x :: \neg x \in S &\lor (\exists y :: y < x :: y \in S)) \)
\end{align*}

= \{ \text{renaming the dummy with (0) and (1)}\}

\begin{align*}
\forall P :: (\forall x :: P x) &\equiv \\
\forall x :: P x &\lor (\exists y :: y < x :: \neg P y))
\end{align*}

= \{ \text{definition of mathematical induction}\}

"mathematical induction over } C \text{ is valid } \\
\text{(End of Proof.)}

\textbf{Theorem 1} "C is well-founded" \equiv (\forall x :: Q x) , \text{ where } Q \text{ is defined as the strongest solution of}

\begin{align*}
Q :: (\forall x :: Q x &\lor (\exists y :: y < x :: \neg Q y)) \)
\end{align*}

\text{Note. The above } Q \text{ is usually interpreted as:}

\begin{align*}
Q x &\equiv \text{"each descending chain starting with } x \\
&\text{is of finite length".}
\end{align*}

\text{The following proof does not appeal to this interpretation. (End of Note.)}

\textbf{Proof} Rewriting (3) as

\begin{align*}
Q :: (\forall x :: (\forall y :: y < x :: Q y) \Rightarrow Q x)
\end{align*}

and recognizing that the antecedent is a monotonic function of } Q \text{, we conclude -Knaster-}

\text{Tarski- that (3) has a strongest solution.}

"C is well-founded" \Rightarrow \{ \text{Theorem 0} \}
\[(\forall x : Qx) \equiv (\forall x : Qx \lor (\exists y : y \prec x : \neg Qy))\]

\[= \{ Q \text{ is a solution of (3)} \} \]

\[(\exists x : Qx) \]

"C is not well-founded"

\[\Rightarrow \{ \text{definition of well-foundedness, (2); for some S} \} \]

\[S \neq \emptyset \land (\forall x : \neg x \in S \lor (\exists y : y \prec x : y \in S))\]

\[= \{ \text{for } P, \text{ as defined by (0), and (1)} \} \]

\[\neg (\forall x : Px) \land (\forall x : \neg Px \lor (\exists y : y \prec x : \neg Py))\]

\[= \{ (3) \} \]

\[\neg (\forall x : Px) \land "P \text{ solves (3)}" \]

\[\Rightarrow \{ Q \text{ is the strongest solution of (3)} \} \]

\[\neg (\exists x : Qx)\]

(End of Proof.)

The above proof was designed in the presence of W.M. Tarski and here recorded under supervision of A.J.M. van Gasteren.

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