Some simple theorems on incremental sorting

A standard component of many in situ sorting algorithms is the operation \( \text{ord} \), given by

\[
\text{ord}.x.y = \begin{cases} 
  x \rightarrow y & \text{if } x > y \\
  \text{skip} & \text{if } x \leq y
\end{cases}
\]

i.e. swapping two values if and only if they are out of order or, in other words, removing an inversion if present. I shall represent the operation \( \text{ord}.x.y \) by

\[
x \rightarrow y
\]

i.e. a dotted arrow from \( x \) to \( y \).

We are interested in other pairs, whose being in order is not destroyed by a collection of \( \text{ord} \) operations; if \( (u,v) \) is such a pair, we shall denote it by

\[
u \rightarrow v
\]

i.e. a drawn arrow from \( u \) to \( v \).

In the following pictorial representation of some theorems there is no aliasing, i.e. different nodes of the graph correspond to different variables.

**Lemma 0** \( u \rightarrow v \& x \rightarrow y \)

**Proof 0** The \( \text{ord} \) operations involve \( x \) and \( y \) only, in particular \( u \) and \( v \) are not involved and their order is maintained. (End of Proof 0.)

**Lemma 1** \( u \rightarrow y \& x \rightarrow v \)
**Proof 1** Operation ord.x.y neither increases x, nor decreases y; from the constancy of u and v the conclusion now follows. (End of Proof 1).

**Lemma 2**

\[
\begin{array}{ccc}
\text{a} & \rightarrow & \text{A} \\
\uparrow & & \uparrow \\
\text{b} & \rightarrow & \text{B}
\end{array}
\]

**Proof 2** Initially, the maximum is an element of \(\{a, A\}\); hence A finally equals the maximum and hence \(B \rightarrow A\) is not destroyed; similarly, the minimum is initially an element of \(\{b, B\}\) and hence finally equal to b, and therefore \(B \rightarrow a\) is not destroyed (End of Proof 2).

Note that it is essential that both ord.a.A and ord.b.B are executed. Lemma 2 is the most interesting of all. The single arrows \(b \rightarrow a\) and \(B \rightarrow A\) can be extended to arbitrary congruent directed graphs, provided ord.x.X is executed between each pair of corresponding variables.

**Lemma 3**

\[
\begin{array}{ccc}
\text{o} & \rightarrow & \text{a} \\
\uparrow & & \uparrow & & \uparrow \\
\text{b} & \rightarrow & \text{o} & \rightarrow & \text{b}
\end{array}
\]

**Proof 3** The value of b is the minimum & the value of a is the maximum. (End of Proof 3.)

As special consequence we mention

**Lemma 4**

\[
\begin{array}{ccc}
\text{o} & \rightarrow & \text{o} \\
\uparrow & & \uparrow \\
\text{b} & \rightarrow & \text{o} \\
\downarrow & & \downarrow \\
\text{o} & \rightarrow & \text{b}
\end{array}
\]

and
From Lemmata 2 and 5 it follows that 4 elements can be sorted by a sequence of 5 ord's:
ord.0.2 & ord.1.3 ; ord.0.1 & ord.2.3 ; ord.1.2
As $2^4 < 4! < 2^5$, 5 ord's is indeed the minimum.

So much for my simple theorems. Needless to say that my feelings about their pictorial representation are very mixed.

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