A methodological remark on mathematical induction

This note is for the record: I know that I have shown the following during several lectures, but cannot remember that I ever wrote it down.

In the following, \( x \) and \( y \) range over the elements of a well-founded set \((S, <)\); \((S, <)\) being well-founded means that for any predicate \( P \) on \( S \)

\[
(\text{Ax}:: P.x) \equiv (\text{Ax}:: P'.x),
\]

where \( P' \) is given in terms of \( P \) by

\[
(\text{Ax}:: P'.x \equiv P.x \lor (\text{Ey}: y<x : \neg P.y))
\]

This is what mathematical induction is about: instead of computing the left-hand side of (0), we can compute its right-hand side, and, if the value is true, the latter computation is "easier" because - see (1) - \( P' \) is formally weaker than \( P \). Indeed, we deduce from (1) immediately

\[
(\text{Ax}:: P.x \Rightarrow P'.x)
\]

\* \* \*

The right-hand side of (0) is of the same form as its left-hand side, so, why don't we make life still easier by repeating the trick, since

\[
(\text{Ax}:: P'.x) \equiv (\text{Ax}:: P''.x)
\]

where according to (1) - with \( P := P' \) -

\[
(\text{Ax}:: P''.x \equiv P'.x \lor (\text{Ey}: y<x : \neg P'.y))
\]

Should we continue, and derive \( P''' \) to make life even easier? No, we can stop at \( P' \), for we observe
for any $x$

$$P'' \cdot x$$

$$= \{(3)\}$$

$$P' \cdot x \lor (Ey: y < x, \neg P' \cdot y)$$

$$= \{(1)\}$$

$$P \cdot x \lor (Ey: y < x, \neg P \cdot y) \lor (Ey: y < x, \neg P' \cdot y)$$

$$= \{\text{predicate calculus}\}$$

$$P \cdot x \lor (Ey: y < x, \neg P \cdot y) \lor (\neg P' \cdot y)$$

$$= \{(2) \text{ and predicate calculus}\}$$

$$P \cdot x \lor (Ey: y < x, \neg P \cdot y)$$

$$= \{(1)\}$$

$$P \cdot x$$

In other words, the decision to demonstrate the truth of $(Ax: P \cdot x)$ by mathematical induction is idempotent: it can be taken once, but then the proof obligation has reached a fixpoint.

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