A bagatelle on implication's transitivity

We recall

- Leibniz's Rule in the form \([X \equiv Y] \Rightarrow [\forall X \equiv \forall Y]\)
- \(\equiv\) is symmetric and associative
- \(\lor\) is symmetric, associative, and idempotent; \(\lor\) distributes over \(\equiv\)
- \(\land\) is defined by the Golden Rule
  \([X \land Y \equiv X \equiv Y \equiv X \lor Y]\)
  it can be shown to be symmetric, associative, and idempotent.

We prove here

\((0)\) \([X \land (Y \equiv Z) \equiv X \land Y \equiv X \land Z \equiv X]\)

**Proof**

\[X \land (Y \equiv Z)\]
\[= \{\text{Golden Rule}\}\]
\[X \equiv Y \equiv Z \equiv X \lor (Y \equiv Z)\]
\[= \{\lor\ \text{distributes over} \equiv\}\]
\[X \equiv Y \equiv Z \equiv X \lor Y \equiv X \lor Z\]
\[= \{\text{sym. \& ass. of} \equiv\}\]
\[(X \equiv Y \equiv X \lor Y) \equiv (Z \equiv X \lor Z)\]
\[= \{\text{Golden Rule, twice}\}\]
\[X \land Y \equiv X \land Z \equiv X\]  \(\text{(End of Proof?)}\)
\[ \Rightarrow \text{ is defined by} \]
\[ [X \Rightarrow Y \equiv X \lor Y \equiv Y] \]
or, by virtue of the Golden Rule, equivalently by
\[ [X \Rightarrow Y \equiv X \land Y \equiv X] \]

The remainder of this note is devoted to proving

\[(1) \quad [(X \Rightarrow Y) \land (Y \Rightarrow Z) \Rightarrow (X \Rightarrow Z)] \quad ,\]

not because this is difficult, but because this proof gives me the opportunity of showing the considerations that guide its design.

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The primary shape of our demonstrandum (1) is of the form \([P \Rightarrow Q]\) as to how to eliminate the \(\Rightarrow\), about which little has been established so far. We can either take the form of the antecedent into account and consider a demonstrandum of the form

\[ [P \land Q \Rightarrow R] \]
or the form of the consequent, and consider a demonstrandum of the form

\[ [P \Rightarrow (Q \Rightarrow R)] \]

We choose the latter because of its nested implications about which we know so little yet: their elimination offers the hope of a "complicated"
expression whose structure almost dictates its simplification. We observe

\[ P \Rightarrow (Q \Rightarrow R) \]

= \{ as the outer implication has a complicated consequent that we would not like to duplicate we opt for its conjunctive definition \}

\[ P \wedge (Q \Rightarrow R) \equiv P \]

= \{ because of the \( \wedge \) in front of it, the remaining implication too is eliminated by its conjunctive definition \}

\[ P \wedge (Q \wedge R \equiv Q) \equiv P \]

= \{ (0) because that tells us how to deal with an equivalence as conjunct: \( (0) \) with \( X, Y, Z := P, Q \wedge R, Q \) \}

\[ P \wedge Q \wedge R \equiv P \wedge Q \]

= \{ conjunctive def. of \( \Rightarrow \) \}

\[ P \wedge Q \Rightarrow R \]

Hence we have established

(2) \[ [ P \Rightarrow (Q \Rightarrow R) \equiv P \wedge Q \Rightarrow R ] \]

Applying this result (2) that tells us how to eliminate a \( \Rightarrow \) from the consequent to our demonstrandum (1), we rewrite that as

(1') \[ [(X \Rightarrow Y) \wedge (Y \Rightarrow Z) \wedge X \Rightarrow Z] \]

So far we have not taken into account that the antecedent of (1) was a conjunction; the
introduction of the conjunct $X$ as in $(1')$ draws attention to this fact: the time has come to take into account that the antecedent of $(1)$ was a conjunction of implications. First and last conjuncts of the antecedent of $(1')$ being the only pair in which only two variables occur, we investigate

\[
P \land (P \Rightarrow Q)
\]

= \{\text{conjunctive elimination of } \Rightarrow, \text{ as before}\}

\[
P \land (P \land Q \equiv P)
\]

= \{(0) with $X,Y,Z := P, P \land Q, P$ : that should give opportunity for simplification!\}

\[
P \land P \land Q \equiv P \land P \equiv P
\]

= \{\text{idempotence of } \land\}

\[
P \land Q \equiv P \equiv P
\]

= \{\text{identity element of } \equiv\}

\[
P \land Q
\]

hence

(3) \[ P \land (P \Rightarrow Q) \equiv P \land Q \]

And now we are ready to tackle our original demonstrandum

(1)

= \{(\text{on account of } (2) : see (1')})

[ $X \land (X \Rightarrow Y) \land (Y \Rightarrow Z) \Rightarrow Z$]

= \{(3) with $P,Q := X,Y$\}
\[ X \land Y \land (Y \implies Z) \implies Z \]
\[
= \{ (3) \text{ with } P, Q := Y, Z \} \]
\[ X \land Y \land Z \implies Z \]
\[
= \{ \text{conjunctive definition of } \implies \} \]
\[ X \land Y \land Z \land Z \equiv X \land Y \land Z \]
\[
= \{ \text{idempotence of } \land \text{ ; identity element of } \equiv \}
\]
true

And this concludes the proof.

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Why should I be interested in heuristics—even in woolly heuristics—to solve a problem so trivial? It is trivial in the technical sense that we have mechanical decision procedures for the propositional calculus. The point is—I hope!—that by willfully ignoring the mechanical decidability I can even use these technically "trivial" problems as proving ground for manipulation strategies.

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