On an exercise of Tony Hoare's

Let $\leq$ be a reflexive, transitive relation;
Let f and g be such that

(0) $\quad f.x \leq y \equiv x \leq g.y \quad$ for all x, y .

Prove that f and g are monotonic with respect to $\leq$ ("preserve $\leq$").

Proof We observe for any u, v

$f.u \leq f.v$

$= \quad \{ \leq$ is reflexive and predicate calculus $\}$

$f.v \leq f.v \Rightarrow f.u \leq f.v$

$= \quad \{ (0)$ with $x, y := v, f.v$ and $x, y := u, f.v \}$

$v \leq g.(f.v) \Rightarrow u \leq g.(f.v)$

$\Leftarrow \quad \{ \leq$ is transitive $\}$

$u \leq v$

Because $\geq$, defined by $x \geq y \equiv y \leq x$, is also reflexive and transitive, the monotonicity of g follows by this symmetry from the above.

(End of Proof)

It is a nice example of a shortest proof in the sense that each of the three steps uses one of our data. A further reason for recording it is that the first step is not a rabbit. Our proof obligation is to remove without weakening the f-applications from $f.u \leq f.v$, and only (0) tells us how to do so, but only for an f-application to the left of $\le$. So it is the f in $f.v$, whose removal presents the problem.

Since we have to use that the f in $f.v$ is the function f that satisfies (0), we <u>have</u> to introduce an expression of the form

$$f.v \le \text{something} \quad .$$

Since we <u>have</u> to use properties of $\le$ as well, the simplest choice for "something" is $f.v$ itself: because $\le$ is reflexive

$$f.v \le f.v$$

equivales $true$. How do we introduce this representation of $true$ without weakening? Without changing the value of a boolean expression, we can prefix it by "$true \equiv$", "$true \wedge$", or "$true \Rightarrow$" ; since of "$a \equiv b$", "$a \wedge b$" and "$a \Rightarrow b$", the last one is the weakest one, we prefix our demonstrandum by

$$f.v \le f.v \Rightarrow \quad .$$

Hence our proof.

Austin, 20 September 1989

prof. dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712-1188
USA