Designing the proof of Vizing's Theorem

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We shall design the proof of the following theorem, due to V.G. Vizing.

**Theorem** For a finite undirected graph without auto-loops and without multiple edges, at any vertex of which fewer than \( N \) edges meet, \( N \) colours suffice for an edge colouring such that edges incident on the same vertex are of different colour.

* * *

The existence of such an edge colouring is demonstrated constructively. To this end it suffices to consider an algorithm that, given an acceptably coloured graph with one uncoloured edge, constructs an acceptably colouring of the whole graph. The graph being acceptably coloured is expressed by

\[(\text{acc.\,V})\]

where predicate \text{acc} on vertices is given by

\[
\text{acc.\,V} \equiv (\text{no two edges incident on vertex V have the same colour})
\]

In the rest of this note we confine ourselves to acceptably coloured graphs. In order not to interrupt the subsequent development of the algorithm
too much, we first introduce a concept, the need of which will emerge, viz. the alternating path. An alternating path is a maximal path of at most two given colours. Because the colouring of the graph is acceptable, those two colours alternate along such a path, hence the name. One can show the following:

(i) an alternating path is either a cycle (i.e. without end points) or a simple path with two end points. (In order to avoid case analysis, we allow the end points to coincide, in which case the alternating path has length 0.)

(ii) a pair of colours and a vertex determine a unique alternating path of those colours and through that vertex. (We allow the two given colours to be equal, in which case the length of the alternating path is at most 1.)

(iii) swapping the colours of the edges of an alternating path maintains (0), i.e. leaves the colouring of the graph acceptable. (Here it is used that an alternating path with end points is maximal, i.e. cannot be extended at its end points.) The importance of swapping the colours of the edges of a noncyclic alternating path is that it changes the set of colours at the end points while leaving the colouring acceptable.

For each vertex \( V \) we define a colour \( c(V) \), in
terms of which we define predicate $\mathcal{P}$ given by
$\mathcal{P}(V) \equiv (\text{no edge incident on } V \text{ has colour } c.V) \quad .$

The fact that the number of colours is higher than the number of edges meeting at any vertex is exploited
by choosing $c$ in such a way that initially ($\mathcal{AV}::\mathcal{P}(V)$)
holds. In the following algorithm — which we explain
afterwards— $X, Y,$ and $Z$ are variables of type
vertex. Pre- and postcondition are given in full;
intermediate assertions are only named and will
be determined later.

$$\{ (\mathcal{AV}::\text{acc}.V) \land (\mathcal{AV}::\mathcal{P}(V)) \land
(\text{XY is the only uncoloured edge}) \}\}
\{P0: \text{invariant}\}
\text{do the c.Y-path ends in } Y \rightarrow \{P1\}
determine Z so that edge
$XZ$ has colour $c.Y$ \{P2\}
; give edge $XY$ colour $c.Y$ and
uncolour edge $XZ$ \{P3\}
; $Y := Z$ \{P0\}
\text{od} \{P4\}
; swap the colours along the c.Y-path \{P5\}
; give edge $XY$ colour $c.Y$
$$\{ (\mathcal{AV}::\text{acc}.V) \land (\text{all edges are coloured}) \}\}

The term
$$(0) \quad (\mathcal{AV}::\text{acc}.V)$$
is a conjunct of all intermediate assertions;
the term

(1)  \((XY \text{ is the only uncoloured edge})\)

is a conjunct of all, except \(P_3\). Note that, in view of the absence of autoloops, (1) implies that \(X\) and \(Y\) are two different vertices.

We now develop the algorithm, beginning with its last statement, viz. colouring \(XY\) with an acceptable colour. The colour is acceptable - i.e. the statement maintains (0) - provided it is incident on neither \(X\) nor \(Y\). Somewhat asymmetrically, we decide to give edge \(XY\) colour \(c.Y\), and take for \(P_5\)

\[ P_5: (0) \land (1) \land (c.Y \text{ not incident on } X) \land f.Y \]

We now consider \(P_5\) as the postcondition to be established. Because the other conjuncts are implied by the program's precondition, we concentrate our attention on establishing

\[ (c.Y \text{ not incident on } X) \]

while maintaining the other three conjuncts of \(P_5\). If an edge incident on \(X\) has colour \(c.Y\), its colour has to be replaced by a colour not incident on \(X\). Since initially \(f.X\), i.e. colour \(c.X\) not incident on \(X\), we propose, in view of (iii), to swap under the initial validity of \(f.X\) the colours along the alternating path through \(X\) and with colours \(c.X\) and \(c.Y\).
We call this "the c.Y-path"; for graphs with 
(NV::acc.V) \land f.X we define - because we need the 
concept a number of times - for any colour p 
the p-path to be the alternating path through X 
with colours p and c.X. Because of f.X, 
colour c.X is not incident on X, and hence 
• no p-path is a cycle 
• the p-path with p = c.X is empty 
• the non-empty p-path starts at X with an 
  edge of colour p and ends at a vertex 
  different from X.

In the above terminology, we proposed to 
establish (c.Y not incident on X) by swapping 
the colours along the c.Y-path; this maintains 
(0) \land (1), but maintains f.Y only provided the 
c.Y-path does not end in Y. (Because of f.Y, 
a c.Y-path ending at Y does so with an edge of 
colour c.X; changing that colour into c.Y would 
falsify f.Y.) Hence the following precondition 
suffices:

P4: (0) \land (1) \land f.X \land f.Y \land 
  (the c.Y-path does not end in Y).

Viewed as postcondition, P4's last conjunct 
comes from the negation of the guard, the others 
have to come from the invariant. As we shall see 
shortly, all is available for the invariance of 
(0) \land (1) \land f.X; the invariance of f.Y, however,
poses a problem. The invariance of \( P \) requires the conjunct \( P_2 \) in \( P_3 \), and the simplest way of justifying it there is by requiring \( P_2 \) as conjunct in \( P_2 \). (Besides this being the simplest way, our termination argument relies on the fact that no false \( P \)-value is truthified; see later.) For the justification of \( P_2 \) in \( P_2 \) we strengthen the invariant with a conjunct that restricts the occurrence of vertices \( V \) with \( P_2 \) as follows:

(2) for any colour \( p \) such that the \( p \)-path is not empty:

- let the edge of colour \( p \) and incident on \( X \) be edge \( XV \);
- let the \( p \)-path end at vertex \( W \);
- then \( c.W = p \Rightarrow P.V \).

Now the time has come to list and justify assertions \( P_0 \) through \( P_3 \).

\[ P_0: (0) \land (1) \land (2) \land P.X \land P.Y \land X \neq Y \]

Assertion \( P_0 \) is implied by the precondition: its first two conjuncts occur in the precondition, the next three follow from \( (AV :: P.V) \), and \( X \neq Y \) follows from (1) and the absence of autoloops. \( P_0 \)'s reestablishment at the end of the repeatable statement will be discussed after \( P_3 \).
$P_1$: $\lnot (the \ c.Y-path \ ends \ in \ Y) \land$
the c.Y-path is at least 2 edges long $\land$
the c.Y-path starts at $X$ with an edge
of colour c.Y $\land$

The first two conjuncts follow from the topology
of the program; the next conjunct follows from
the c.Y-path being a coloured connection between
$X$ and $Y$, and hence different from the unique
edge $XY$, which is uncoloured; the last conjunct
is a property of nonempty p-paths $\land$

$P_2$: $P_1 \land (XZ \ has \ colour \ c.Y) \land X \neq Z \land Y \neq Z \land$
$\neg Z \land \neg C.Z \land c.Y$ $\land$

The first conjunct $P_1$, which does not refer to
$Z$, is maintained and implies that $Z$ is properly
defined; $XZ$ is the first edge on the c.Y-path and hence
has colour c.Y; $XZ$ being an edge, $X \neq Z$; the
path length being $\geq 2$, $Y \neq Z$. The conjunct $\neg Z$
follows from (2) with the instantiation
$p, v, w := c.Y, Z, Y$. From $\neg Z$ and the fact that
$XZ$ has colour c.Y, we conclude $C.Z \neq c.Y$ (from
which $Z \neq Y$ could have been concluded).

$P_3$: $(0) \land (XZ \ is \ the \ only \ uncoloured \ edge) \land$
(XY has colour c.Y) $\land p.X \land \exists p.Y \land \exists Z \land (2)$

We now justify the conjuncts of $P_3$ in order.

For (0) we consider $X, Y, Z$ since for all other
vertices $V$, acc.$V$ remains unchanged; acc.$X$ re-
mains unchanged because the bag of colours incident on $X$ is unchanged; acc.$Y$ is not falsified on account of $f.Y$ in $P_2$; acc.$Z$ is not falsified since the bag of colours incident on $Z$ is decreased.

The next two conjuncts follow from (1), the statement, and the fact that $XY$ and $XZ$ are different edges.

The term $f.X$ is maintained since the bag of colours incident on $X$ is left unchanged; $f.Y$ is falsified because colour $c.Y$ is given to an edge incident on $Y$; $f.Z$ is maintained because the bag of colours incident on $Z$ is decreased.

For the invariance of (2), we distinguish two cases.

In the case $p \neq c.Y$ we observe that the colouring of $XY$ and the uncolouring of $XZ$ leaves the $p$-path unchanged; in particular, if the $p$-path is not empty, its $V$ and $W$ remain the same and, because of the absence of multiple edges, $V \neq Y$. Since $c$ is a constant function and $Y$ is the only vertex whose $f$-value changes, the invariance of

$$c.W = p \Rightarrow f.V$$

for $p \neq c.Y$ follows.

In the case $p = c.Y$ we observe that the $c.Y$-path $XZ...Y$ is replaced by the $c.Y$-path
XY...Z, i.e. we have to demonstrate

\[ c.W = p \Rightarrow f.V \]

for the instantiation \( p, V, W := c.Y, Y, Z \). We observe

\[
\begin{align*}
  c.W &= p \\
  &= \{ p, W := c.Y, Z \} \\
  c.Z &= c.Y \\
  &= \{ P2 \} \\
  &= \text{false} \\
  \Rightarrow &\{ \text{predicate calculus} \} \\
  &= f.V
\end{align*}
\]

The verification of \([P3 \Rightarrow \text{wp.""Y:=Z"".P0}]\) is left to the reader.

For the termination argument we observe that, the function \( c \) being constant, \( f \)-values can only be changed by changing edge colours. The (un)colour statement in the repetition decreases \((NV::f.V)\) by 1; hence the repetition terminates.

And this concludes our proof of Vizing's Theorem.

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In retrospect

The above is our second effort and the result is much better than our first presentation (which was probably still too much influenced by the published proofs).

The algorithm is more subtle than it reveals at first sight. It is clear that the repeatable statement has to change the uncoloured edge, and once it has been decided to uncolour $XZ$, the text of the repeatable statement follows. The more subtle point is that at the beginning of the repeatable statement, the situation seems symmetric in $X$ and $Y$: the alternating path with colours $c.X$ and $c.Y$ through $X$ ends at $Y$, but the alternating path of those colours through $Y$ ends at $X$! The symmetry is destroyed by invariant (2) via the definition of the $p$-path, and while $Y$ is a variable, $X$ is a constant, and the set of $p$-paths is a linear collection.

The crux of the argument is, of course, conjunct (2) of the invariant. Again, (2) is in its formulation more subtle than might be appreciated at first sight. Its mathematical contents is easily established: strong enough to allow
the inference of $f,Z$ in $P_2$, weak enough to be maintained: $(AV_2: f,V)$ would be too strong. It was the formulation of (2) that required careful design. The one-point rule makes it possible to simplify the formulation by eliminating the dummy $p$, but this simplification of (2) makes the case analysis in its proof of invariance much harder to capture.

Finally we observe with satisfaction that we needed only very few variables (viz. $X, Y,$ and $Z$, of which $X$ is constant and $Z$ in essence a variable local to the repeatable statement).

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