Well-foundedness and the relational calculus

In a well-founded set, all decreasing chains are of finite length. In general, we cannot guarantee finite lengths for increasing chains, i.e. the transpose of a well-founded relation is, in general, not well-founded. Yet, there is something special about that transpose: its transpose is well-founded! This suggests the introduction of two forms of well-foundedness, which we may call "left-founded" and "right-founded", with the general connection

\[(S \text{ is left-founded}) \equiv (\sim S \text{ is right-founded})\]  

For "S is left-founded" we propose the definition -P and S being of type relation-

\[(0) \quad (\forall P : [P] \iff [P \lor S; \neg P]) \quad \text{(See Appendix.)}\]

**Legend:** We follow the convention that seems in the process of being established of giving ";" a binding power between the unary "\(\neg\)" and "\(\sim\)" on the one hand and the binary "\(\lor\)" and "\(\land\)" on the other. (End of Legenda.)

For "S is right-founded" we propose

\[(\forall P : [P] \iff [P \lor \neg P; S])\]

The proof of the general connection is left as an exercise for the reader, who needs

- \([X] \equiv [\sim X]\)
- \([X \equiv \sim \sim X]\)
\begin{itemize}
  \item \( \sim (X;Y) \equiv \sim Y; \sim X \)
  \item \( \sim \) distributes over boolean operators (in particular over \( \lor \) and \( \land \)).
\end{itemize}

In the sequel, we confine our attention to the notion of left-foundedness.

\* \* \* \* \*

We define the transitive closure of \( S \) as the strongest \( R \) satisfying

\[(1) \quad [R \equiv S \lor S; R]\]

\text{Remark} \hspace{1em} \text{Alternatively, one can replace, in the above definition, (1) by} \hspace{1em} \text{\( [R \equiv S \lor R; S] \)}

\text{or by} \hspace{1em} \text{\( [R \equiv S \lor R; R] \). The equivalence of these three definitions of the transitive closure of \( S \) falls outside the scope of this note. A consequence of this equivalence is that the transitive closure of the transpose of a relation equals the transpose of its transitive closure. (End of Remark.)}

\text{Theorem} \hspace{1em} \text{For left-founded} \( S \), (1) determines \( R \) uniquely.

\text{Proof.} \hspace{1em} \text{With} \hspace{1em} U \text{ satisfying}

\[(2) \quad [U \equiv S \lor S; U]\]

\text{we have to show that} \hspace{1em} [U \equiv R] \text{ follows from (0), (1), (2). We observe}
\[ U = R \]
\[ \{ (0) \text{ with } P := U = R : S \text{ is left-founded} \} \]
\[ \{ (U = R) \lor S; (U \neq R) \} \]
\[ \{ (1), (2) \} \]
\[ \{ v \text{ distributes over } \equiv \} \]
\[ \{ S \lor (S; U = S; R) \lor S; (U \neq R) \} \]
\[ \{ v \text{ distributes over } \equiv \} \]
\[ \{ ; \text{ distributes over } v \} \]
\[ \{ S \lor (S; (U \lor (U \neq R)) \equiv S; (R \lor (U \neq R))) \} \]
\[ \{ \text{pred. calc.} : [X \lor (X \neq Y) \equiv X \lor Y] \} \]
\[ \{ S \lor (S; (U \lor R) \equiv S; (U \lor R)) \} \]
\[ \text{true} . \quad \text{(End of Proof.)} \]

**Theorem 1** The transitive closure of a left-founded relation is left-founded.

**Proof** With \( S \) and \( R \) satisfying (0) and (1) we have to show

(3) \( \langle \forall P : [P] \subseteq [P \lor R; \neg P] \rangle \) .

To this end we first observe for any \( P \)

\[ R; \neg P \]
\[ = \{ 1 \} \]
\[ (S \lor S; R); \neg P \]
\[ = \{ ; \text{ distributes over } v \} \]
\[ S; \neg P \lor S; R; \neg P \]
\{\text{distributes over } \lor\}\}
\begin{align*}
&= S; (\neg P \lor R; \neg P) \\
&\quad \{\text{de Morgan}\}
\end{align*}
\begin{align*}
&= S; \neg (P \land \neg (R; \neg P)) \\
&\quad \text{i.e. we have used (1) to establish}
\end{align*}
\begin{align*}
\text{(4)}& \quad [R; \neg P \equiv S; \neg (P \land \neg (R; \neg P))] \\
\text{This formula relates an } R; \neg P \text{ to an } S; \neg P \text{. We now proceed}
\end{align*}
\begin{align*}
&= [P \lor R; \neg P] \\
&\quad \{\text{pred. calc., to strengthen the induction hypothesis}\}
\end{align*}
\begin{align*}
&= [(P \land \neg (R; \neg P)) \lor R; \neg P] \\
&\quad \{(4)\}
\end{align*}
\begin{align*}
&= [(P \land \neg (R; \neg P)) \lor S; \neg (P \land \neg (R; \neg P))] \\
&\quad \{(0) \text{ with } P := P \land \neg (R; \neg P)\}
\end{align*}
\begin{align*}
&\Rightarrow [P \land \neg (R; \neg P)] \\
&\quad \{\text{pred. calc.}\}
\end{align*}
\begin{align*}
&\Rightarrow [P] \quad \text{(End of Proof?)}
\end{align*}

\textbf{Theorem 2} \quad \text{If the transitive closure of a relation is left-founded, so is the relation itself.}

\textbf{Proof} \quad \text{We have to establish (0) on account of (1) and (3). To this end we observe for any } P
\begin{align*}
&[P \lor S; \neg P]
\end{align*}
\[ \Rightarrow \{ (1) \}, \text{hence } [S \Rightarrow R], \text{and monotonicity} \]
\[ [P \land R; \neg P] \]
\[ \Rightarrow \{ (3) \} \]
\[ [P] \]

(End of Proof?)

* * *

The above theorems and proofs are essentially the same as those in AvG88/EWD1079 "Well-foundedness and the transitive closure" from 1990.04.28. (The identifiers R and S have exchanged roles. I am sorry about that.) I have wanted for a long time to give these proofs as rendered here, but did not succeed because in (0) I had restricted the range of dummy P to left-conditions:

\[ \langle \forall P: [P; \text{true } \equiv P]: \ldots \ldots \ldots \ldots \ldots \rangle \]

Last weekend, looking again at the problem, I recovered from this mistake.

I am fascinated by the above proofs because they are carried out in our "pointless logic": no need for "point predicates" or "line relations"! And that is very nice if we contrast that to the other... and I am afraid typical way of defining well-foundedness.

This is done either by stating that each nonempty subset has a minimal element or by stating that all
decreasing chains are of finite length. Both formulations most explicitly refer to the individual elements. In (0), the third characterization of well-foundedness — viz. the validity of proofs by mathematical induction — is stated as is the notion of transitive closure in (1) in the pointless relational calculus. It is now clear why the third characterization of well-foundedness is to be preferred: whatever can be achieved without postulating "points" is more simply done without them.

I am very pleased with the above results.

Appendix

Formula (0) is the "pointless" transcription of

\[(5) \langle \forall P. \langle \forall x, y :: xPy \rangle \iff \\
\langle \forall x, y :: xPy \lor \langle \exists z :: xSz \land \neg zPy \rangle \rangle \].

In this appendix, we shall show that (5) is equivalent with (6), the traditional way of expressing that \( S \) is a well-founded relation:

\[(6) \langle \forall Q. \langle \forall x :: Qx \rangle \iff \\
\langle \forall x :: Qx \lor \langle \exists z :: xSz \land \neg Qz \rangle \rangle \].

(Usually "\( xSz \)" is rendered as "\( x \succ z \)" or "\( x \in z \)."

Our proof is by mutual implication.
\[(6) \iff (5) \]
\[(6) \]
\[
\{ \text{quantification over a fresh dummy with a nonempty range is the identity operation?} \}
\]
\[
\langle \forall Q :: \langle \forall x,y :: Q \cdot x \rangle \iff \\
\langle \forall x,y :: Q \cdot x \lor \langle \exists z :: x \cdot S \cdot z \land \neg Q \cdot z \rangle \rangle \}
\]
\[
\{ \text{a predicate Q corresponds to a relation P that does not depend on the other argument. By extending the range for P to all relations, the universal quantification is strengthened} \}
\]
\[(5) \]

\[(6) \Rightarrow (5) \]
\[(6) \]
\[
\{ \text{write } Q \cdot x \text{ as } x \cdot P \cdot y ; \ "\forall Q" \text{ then becomes } "\forall P, y" \}
\]
\[
\langle \forall P :: \langle \forall y :: \langle \forall x :: x \cdot P \cdot y \rangle \iff \\
\langle \forall x :: x \cdot P \cdot y \lor \langle \exists z :: x \cdot S \cdot z \land \neg z \cdot P \cdot y \rangle \rangle \rangle \}
\]
\[
\{ \text{monotonicity of } \forall \}
\]
\[
(5) \}
\]

(End of Appendix).

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