On a formula designed by Henk Doornbos

We are asked to prove for any \( x, y, z \)

\[(0) \quad [x; y \land z \Rightarrow (z; \neg y \land x); (\neg x; z \land y)] \]

We don't know beforehand how much of the relational calculus we shall need, but it is almost certain that we shall need the exchange rules

\[
[x; y \Rightarrow z] \equiv [\neg z; \neg y \Rightarrow \neg x] \\
[x; y \Rightarrow z] \equiv [\neg x; \neg z \Rightarrow \neg y]
\]

as they embody the axiom connecting the transposition \( \neg \) and the composition \( ; \).

In a moment we shall need that composition is monotonic; we may need that composition is universally disjunctive (from which the monotonicity follows).

In (0), it is the consequent that needs to be unravelled, because nothing nice is known about the compositions of conjunctions. So we replace the consequent by \( r; s \) and since

\[(1) \quad [x; y \land z \Rightarrow r; s] \]

is a monotonic function of \( r \) and \( s \), we can apply the theorem of EWD1117.
Because, in view of the exchange rule, the expressions replaced are quite acceptable as antecedents, we use from EWD1117 that
for monotonic $\Rightarrow$ 

$$[f; \text{exp}] \equiv (\forall z: [\text{exp} \Rightarrow z]; [f;z])$$

i.e. we propose to prove (0) by showing that for any $r, s, (1)$ follows from

(2a) \hspace{1em} [z; \neg y \land x \Rightarrow r] \\
(2b) \hspace{1em} [\neg x; z \land y \Rightarrow s]

Because $z$ occurs fairly isolated in (1) we try to isolate $z$ from (2). We observe

(2) 
= \{definition\} 
= [z; \neg y \land x \Rightarrow r] \land [\neg x; z \land y \Rightarrow s] 
= \{shunting in preparation for exchanges\} 
= [z; \neg y \Rightarrow \neg x \lor r] \land [\neg x; z \Rightarrow \neg y \lor s] 
= \{exchange rules and de Morgan\} 
= [ (x \land \neg r); y \Rightarrow \neg z] \land [ x; (y \land s) \Rightarrow \neg z] 
= \{predicate calculus\} 
= [ (x \land \neg r); y \lor x; (y \land s) \Rightarrow \neg z]

Using the above rewriting of (2) we observe

(1) 
= \{definition\}
\[ [x:y \wedge z \Rightarrow r;s] = \{ \text{shunting} \} [x:y \Rightarrow r;s \lor rz] \]
\[ \Leftarrow \{ (2) \text{ as rewritten} \} [x:y \Rightarrow r;s \lor (x \land rz);y \lor x;(y \land zs)] = \{ \text{see below} \} \text{true} \]

Our final obligation is to weaken \( x;y \)

to the above disjunction, at the same time

introducing \( r,s \), and the conjuncts \( rz, zs \).

We observe to this end

\[ x;y = \{ [y \equiv (y \land s) \lor (y \land zs)], "; is disjunctive\} x;(y \land s) \lor x;(y \land zs) \]
\[ = \{ [x \equiv (x \land r) \lor (x \land rz)], "; is disjunctive? \} (x \land r);(y \land s) \lor (x \land rz);(y \land s) \lor x;(y \land zs) \]
\[ \Rightarrow \{ ";" is monotonic\} r;s \lor (x \land rz);y \lor x;(y \land zs) \]

QED

Remark I met the formula in the form
\((1) \Leftarrow (2)\), in which form it was attributed to

Henk Doornbos (and called the Grand Dedekind rule).

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prof. dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78750-1188, USA