A summary for Turner’s class (An extract from EWD1107)

Let $\equiv$ - read “below” - be a punctual relation on structures of some type, i.e. for all $u, v, x, y$

$$[ u = v \land x = y \Rightarrow (u \equiv x = v \equiv y)]$$

For $\equiv$ a preorder - i.e. a reflexive and transitive relation -

(0) $$[ x \equiv y \equiv (\forall z:: z \equiv x \Rightarrow z \equiv y)]$$ and

(1) $$[ x \equiv y \equiv (\forall z:: x \equiv z \Leftarrow y \equiv z)]$$

For a preorder $\equiv$, a value $k$ satisfying

(2) $$\langle \forall x:: [ x \equiv k \equiv (\forall y: y \in W: x \equiv f(y)) \rangle \rangle$$

is called “a highest lower bound” for the set $S$ given by

$$S = \{ z | (\exists y: y \in W: f(y) \}$$

(Note that, for $f$ the identity function, $S = W$.)

Moreover, a value $h$ satisfying

(3) $$\langle \forall x:: [ h \equiv x \equiv (\forall y: y \in W: f(y) \equiv x) \rangle \rangle$$

is called “a lowest higher bound for $S$”. We remind the reader that these two names only make sense because we have chosen to read the symbol $\equiv$ as “below”. For the rest of this note we restrict ourselves to a preorder $\equiv$ such that highest lower and lowest higher bounds exist for any set $S$. Each of these bounds is unique if $\equiv$ is antisymmetric as well, i.e.
\[ [x \leq y \land y \leq x \Rightarrow x = y] ; \]

a preorder that is antisymmetric as well is called "a partial order" and in what follows, \( \leq \) is further restricted to a partial order. We now show the uniqueness of the highest lower bound.

Proof: Let \( k' \) satisfy

\[(2') \quad \langle \forall x :: [x \in k' \equiv \langle \forall y : y \in W : x \in f(y) \rangle] \rangle \]

We then observe

\[
\text{true} = \{ (2) \text{ and } (2') \}\]
\[
\langle \forall x :: [x \in k \equiv x \in k'] \rangle \Rightarrow \{ x := k \text{ and } x := k' \}
\]
\[
[ k \leq k \equiv k \leq k' ] \land [ k \leq k' \equiv k' \leq k ]
\]
\[
= \{ \geq \text{ is reflexive} \}
\]
\[
[ k \leq k' ] \land [ k' \leq k ]
\]
\[
\Rightarrow \{ \geq \text{ is antisymmetric} \}
\]
\[
[ k = k' ]
\]  

(End of Proof.)

The highest lower bound of the empty set is traditionally denoted by \( T \) and called "top"; the lowest higher bound of the empty set is traditionally denoted by \( \bot \) and called "bottom". From (2) and (3) follow

\[
\langle \forall x :: [x \in T] \rangle \text{ and } \langle \forall x :: [\bot \in x] \rangle .
\]
The unique value for \( k \) satisfying (2) is denoted by \(< Ny: y \in W: f_y >\), i.e.

\[(4) \langle \forall x : [ x \in \langle Ny: y \in W: f_y > \equiv \langle Ny: y \in W: x \in f_y \rangle \rangle \rangle .\]

Similarly, the lowest higher bound is denoted using \( \uparrow \):

\[(5) \langle \exists x : [ \langle \uparrow y : y \in W: f_y \rangle \leq x \equiv \langle \forall y : y \in W: f_y \leq x \rangle ] \rangle .\]

* * *

Function \( f \) is monotonic means here

\[(6) \ [ x \leq y ] \Rightarrow [ f_x \leq f_y ] \quad \text{for all } x, y .\]

For monotonic \( f \) we can prove (7) and (8):

\[(7) \ [ f_{\langle Ny: y \in W: y >} \leq \langle Ny: y \in W: f_y > \rangle \]

\[(8) \ [ \langle \uparrow y : y \in W: f_y > \leq f_{\langle \uparrow y : y \in W: y >} \rangle ,\]

of which we prove (7).

Proof In the following we leave the ranges \( x \in W, y \in W \)
understood and observe for any monotonic \( f \)

\[
[f_{\langle Nx:: x >} \leq \langle Ny:: f_y >] \]

\[= \{ (4) \text{ with } x := f_{\langle Nx:: x >} \} \]

\[\langle \forall y : f_{\langle Nx:: x >} \leq f_y > \rangle \]

\[= \{ \text{interchange} \} \]

\[\langle \forall y : [ f_{\langle Nx:: x >} \leq f_y ] \rangle \]

\[\Leftrightarrow \{ f \text{ is monotonic; predicate calculus} \} \]

\[\langle \forall y : [ \langle Nx:: x > \leq y ] \rangle \]

\[= \{ (4) \text{ with } x, f := \langle Nx:: x >, \text{identity} \} \]

\[\langle \langle Nx:: x > \equiv \langle Ny:: y > \rangle \]
\[ \{ E \text{ is reflexive} \} \]

true.

(End of Proof.

* * *

We are now ready for the theorem of Knaster-Tarski: for monotonic \( f \), the equations
\[ (9) \quad x: [f.x \leq x] \]
and
\[ (10) \quad x: [f.x = x] \]
have each a lowest solution, and the two lowest solutions are the same.

Proof We define \( q \) by
\[ [q = \langle \Pi y: [f y \leq y]: y \rangle] \]
and observe, in order to show that \( q \) solves (9)
\[ f.q \]
\[ = \{ \text{definition of } q \} \]
\[ = \langle \Pi y: [f y \leq y]: y \rangle \]
\[ = \{ f \text{ is monotonic, (7)} \} \]
\[ = \langle \Pi y: [f y \leq y]: f y \rangle \]
\[ = \{ \Pi \text{ is monotonic, see (12) below} \} \]
\[ = \langle \Pi y: [f y \leq y]: y \rangle \]
\[ = \{ \text{definition of } q \} \]
\[ = q \]

which observation establishes \( [f.q \leq q] \) thanks to the transitivity of \( E \); so \( q \) solves (9), and it is its lowest solution by construction, for we observe
\[(11) \quad \langle \forall y : [f y \leq y] : q_5 y \rangle \]
\[= \quad \{(4) \text{, def of } \Pi \}
\[q \in \langle \forall y : [f y \leq y] : y \rangle \]
\[= \quad \{ \text{def of } q \text{, } \leq \text{ is reflexive} \}
\[\quad \text{true.} \]

Next we observe

\[\text{[} f \cdot q = q \text{]} \]
\[\Rightarrow \quad \{ \text{ } \leq \text{ is antisymmetric} \}
\[\text{[} f \cdot q \leq q \text{]} \land \text{[} q \leq f \cdot q \text{]} \]
\[\Rightarrow \quad \{(11) \text{ with } y := f \cdot q \}
\[\text{[} f \cdot q \leq q \text{]} \land \text{[} f \cdot (f \cdot q) \leq f \cdot q \text{]} \]
\[\Rightarrow \quad \{ f \text{ is monotonic} \}
\[\text{[} f \cdot q \leq q \text{]} \land \text{[} f \cdot q \leq q \text{]} \]
\[\Rightarrow \quad \{ q \text{ solves } (9) \}
\[\quad \text{true} \quad , \]

so \(q\) solves \((10)\) as well. Moreover, \(q\) is \((10)'s\) lowest solution, as follows from the observation

\[\langle \forall y : [f y = y] : q \leq y \rangle \]
\[\Rightarrow \quad \{ [f y = y] \Rightarrow [f y \leq y] \text{ since } \leq \text{ is reflexive} \}
\[\langle \forall y : [f y \leq y] : q \leq y \rangle \]
\[\Rightarrow \quad \{(11)\}
\[\quad \text{true.} \]

* * *

The monotonicity of \(\Pi\) is expressed by

\[(12) \quad [\langle \forall y : g y \leq h y \rangle \Rightarrow \langle \forall y : g y \rangle \leq \langle \forall y : h y \rangle] \]
where we left the range of \( y \) understood.

Proof. We observe for any \( g, h, \) and range of \( y \)

\[
\langle \forall y :: g.y \rangle \subseteq \langle \forall y :: h.y \rangle
\]

\[
= \{ \text{(0)} \}
\]

\[
\langle \forall z :: z \in \langle \forall y :: g.y \rangle \Rightarrow z \in \langle \forall y :: h.y \rangle \rangle
\]

\[
= \{ \text{(4) twice} \}
\]

\[
\langle \forall z :: \langle \forall y :: z \in g.y \rangle \Rightarrow \langle \forall y :: z \in h.y \rangle \rangle
\]

\[
\Leftrightarrow \{ \text{monotonicity of } \forall \}
\]

\[
\langle \forall z :: \langle \forall y :: z \in g.y \Rightarrow z \in h.y \rangle \rangle
\]

\[
= \{ \text{interchange} \}
\]

\[
\langle \forall y :: \langle \forall z :: z \in g.y \Rightarrow z \in h.y \rangle \rangle
\]

\[
= \{ \text{(0)} \}
\]

\[
\langle \forall y :: g.y \subseteq h.y \rangle \quad \quad \text{(End of Proof)}
\]

Notice how, in the above proof, we started using (0) and not (1), which would have introduced \( \forall \) to the left of \( \subseteq \), whereas (4) enables us to eliminate \( \forall \) to the right of \( \subseteq \).

Austin, 12 February 1992

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