Total-order junctivity

Continuity - see [0], p.87 - is a special case of total-order junctivity. A set $S^*$ of predicate transformers is totally ordered iff

$$\forall x, y \in S \land y \in S: [x \Rightarrow y] \lor [x \Leftarrow y]$$

and a predicate transformer is called "total-order junctive" iff it is junctive over totally ordered sets of predicates. We leave to the reader the proof of - see [0], p.84 -

$S$ is totally ordered $\equiv S^*$ is totally ordered ,

and also of "any total-order junctive predicate transformer is monotonic". We now generalize Theorem (6,43) of [0] - see p.99 - to

Theorem Disjunction preserves total-order conjunctivity, i.e. let predicate transformer $f$ be given in terms of the total-order conjunctive predicate transformers $g$ and $h$ by

$$[f.x = g.x \lor h.x] \text{ for all } x$$

then $f$ is total-order conjunctive.

Proof In this proof, $x$ and $y$ range over an arbitrary totally ordered set. Our proof obligation is then to show

$$[f.(\forall x::x) = (\forall x::f.x)]$$
To this end we observe

\[ f \cdot \langle \forall x :: x \rangle \]
\[ = \{ \text{def. of } f \text{ and renaming a dummy} \} \]
\[ g \cdot \langle \forall x :: x \rangle \lor h \cdot \langle \forall y :: y \rangle \]
\[ = \{ g \text{ and } h \text{ total-order conjunctive} \} \]
\[ \langle \forall x :: g \cdot x \rangle \lor \langle \forall y :: h \cdot y \rangle \]
\[ = \{ \text{distribute } \lor \text{ over } \forall ; \text{ unnesting} \} \]
\[ \langle \forall x, y :: g \cdot x \lor h \cdot y \rangle \]
\[ = \{ \text{range is totally ordered} \} \]
\[ \langle \forall x, y :: [x \Rightarrow y] : g \cdot x \lor h \cdot y \rangle \land \]
\[ \langle \forall x, y :: [x \Leftarrow y] : g \cdot x \lor h \cdot y \rangle \]
\[ = \{ \text{nesting} \} \]
\[ \langle \forall x :: \langle \forall y :: [x \Rightarrow y] : g \cdot x \lor h \cdot y \rangle \rangle \land \]
\[ \langle \forall y :: \langle \forall x :: [x \Leftarrow y] : g \cdot x \lor h \cdot y \rangle \rangle \]
\[ = \{ h \text{ monotonic}; g \text{ monotonic} \} \]
\[ \langle \forall x :: g \cdot x \lor h \cdot x \rangle \land \langle \forall y :: g \cdot y \lor h \cdot y \rangle \]
\[ = \{ \text{def. of } f \} \]
\[ \langle \forall x :: f \cdot x \rangle \]  

(End of Proof)

The analog of Theorem (5, 116) of [O] - see p. 76-77 - is:

With \( f \) monotonic in both arguments and \( x \) and \( y \) ranging over some totally ordered set

\[ \langle \forall x, y :: f \cdot x \cdot y \rangle \equiv \langle \forall x :: f \cdot x \cdot x \rangle \]

which we could have appealed to in the above proof. All this is more general, cleaner.
and simpler than in [0]. For the sake of completeness (and having just started a new page) we prove the last theorem by observing

\[\forall x, y :: f(x, y)\]
\[= \{\text{range is totally ordered}\}\]
\[\forall x, y :: [x \Rightarrow y] : f(x, y) \land \forall x, y :: [x \Leftarrow y] : f(x, y)\]
\[= \{\text{nesting}\}\]
\[\forall x :: \forall y :: [x \Rightarrow y] : f(x, y) \land \forall y :: \forall x :: [x \Leftarrow y] : f(x, y)\]
\[= \{\text{f monotonic in both arguments}\}\]
\[\forall x :: f(x, x) \land \forall y :: f(y, y)\]
\[= \{\text{pred. calc.}\}\]
\[\forall x :: f(x, x)\].

All this has been triggered by a theorem in the thesis of Ernie Cohen.

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