Ping-pong arguments and Leibniz's principle

Let $\downarrow$ denote an idempotent, symmetric, and associative infix operator; it is given a higher binding power than the relational operators.

In terms of $\downarrow$ we define a relation $\leq$ by postulating for any x, y

(0) $\qquad x \leq y \;\equiv\; x\downarrow y = x \qquad .$

We would like to prove for any w, x, y

(1) $\qquad w \leq x\downarrow y \;\equiv\; w \leq x \;\wedge\; w \leq y \qquad .$

Applying (0) three times to eliminate the $\leq$s, we rewrite demonstrandum (1) as

(2) $\qquad w\downarrow x\downarrow y = w \;\equiv\; w\downarrow x = w \;\wedge\; w\downarrow y = w \qquad .$

We use Leibniz's Principle in the form

$$a = b \;\wedge\; f.a \;\Rightarrow\; f.b$$

and proceed to prove (2) by a ping-pong argument. For pong we observe for any w, x, y

$$\begin{array}{ll} & w\downarrow x = w \;\wedge\; w\downarrow y = w \\ \Rightarrow & \quad \{\text{Leibniz}\} \\ & (w\downarrow x)\downarrow y = w \\ = & \quad \{\downarrow \text{ associative}\} \\ & w\downarrow x\downarrow y = w \qquad . \end{array}$$

For ping we only prove (for brevity's sake) $w{\downarrow}x{\downarrow}y = w \Rightarrow w{\downarrow}x = w$ by observing

$$
\begin{array}{ll}
 & w{\downarrow}x{\downarrow}y = w \\
= & \quad \{\text{predicate calculus}\} \\
 & w{\downarrow}x{\downarrow}y = w \ \wedge \ w{\downarrow}x{\downarrow}y = w \\
= & \quad \{\downarrow \text{ idempotent}\} \\
 & w{\downarrow}x{\downarrow}y = w \ \wedge \ w{\downarrow}(x{\downarrow}x){\downarrow}y = w \\
= & \quad \{\downarrow \text{ symmetric and associative}\} \\
 & w{\downarrow}x{\downarrow}y = w \ \wedge \ (w{\downarrow}x{\downarrow}y){\downarrow}x = w \\
\Rightarrow & \quad \{\text{Leibniz}\} \\
 & w{\downarrow}x = w \quad .
\end{array}
$$

It was a pleasant surprise to see that (1) had a simpler proof via (2) than via all sorts of other properties of $\leq$. The reason for recording the proof, however, is that its structure reveals Leibniz's Principle as a generator of ping-pong arguments. (Moreover, for a lecture on Leibniz's Principle, it is a very nice example.)

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