Ping-pong arguments and Leibniz's principle

Let $\downarrow$ denote an idempotent, symmetric, and associative infix operator; it is given a higher binding power than the relational operators.

In terms of $\downarrow$ we define a relation $\preceq$ by postulating for any $x, y$

$$(0) \quad x \preceq y \equiv x \downarrow y = x \quad .$$

We would like to prove for any $w, x, y$

$$(1) \quad w \preceq x \downarrow y \equiv w \preceq x \land w \preceq y \quad .$$

Applying (0) three times to eliminate the $\preceq$s, we rewrite demonstrandum (1) as

$$(2) \quad w \downarrow x \downarrow y = w \equiv w \downarrow x = w \land w \downarrow y = w \quad .$$

We use Leibniz's Principle in the form

$$a = b \land f. a \Rightarrow f. b$$

and proceed to prove (2) by a ping-pong argument. For pong we observe for any $w, x, y$

$$w \downarrow x = w \land w \downarrow y = w \Rightarrow$$

$$\{ \text{Leibniz}\}$$

$$(w \downarrow x) \downarrow y = w$$

$$= \{ \downarrow \text{associative}\}$$

$$w \downarrow x \downarrow y = w \quad .$$
For ping we only prove (for brevity's sake) \( w \downarrow x \downarrow y = w \Rightarrow w \downarrow x = w \) by observing

\[
\begin{align*}
    w \downarrow x \downarrow y &= w \\
    &= \{ \text{predicate calculus} \} \\
    w \downarrow x \downarrow y &= w \land w \downarrow x \downarrow y = w \\
    &= \{ \downarrow \text{ idempotent} \} \\
    w \downarrow x \downarrow y &= w \land w \downarrow (x \downarrow x) \downarrow y = w \\
    &= \{ \downarrow \text{ symmetric and associative} \} \\
    w \downarrow x \downarrow y &= w \land (w \downarrow x \downarrow y) \downarrow x = w \\
    \Downarrow & \{ \text{Leibniz} \} \\
    w \downarrow x &= w
\end{align*}
\]

It was a pleasant surprise to see that (1) had a simpler proof via (2) than via all sorts of other properties of \( \preceq \). The reason for recording the proof, however, is that its structure reveals Leibniz's Principle as a generator of ping-pong arguments. (Moreover, for a lecture on Leibniz's Principle, it is a very nice example.)

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