Another ping-pong argument from Leibniz's principle

Here is -see EWD 1186- another very simple proof in which our obligation to use Leibniz's principle leads to a ping-pong argument. The proof comes from the beginning of lattice theory.

Of infix operators \( \uparrow \) ("up") and \( \downarrow \) ("down") and \( x \) and \( y \), we are given

(0) \[ x \uparrow (y \downarrow x) = x \]

(1) \[ y \downarrow (x \uparrow y) = y \]

The reader is more or less supposed to recognize the Absorption Laws. Please note that (0) and (1) are transformed into each other by the interchange \[ x, \uparrow \leftrightarrow y, \downarrow \].

We are asked to prove

(2) \[ x \uparrow y = x \Leftrightarrow y \downarrow x = y \]

Please note that the two sides of this equivalence are transformed into each other by that same interchange \[ x, \uparrow \leftrightarrow y, \downarrow \].

Ping - i.e. \[ x \uparrow y = x \Rightarrow y \downarrow x = y \] - is proved by observing
\[ y \uparrow x \]
\[ = \{ \text{LHS, i.e. } x \uparrow y = x \} \]
\[ y \downarrow (x \uparrow y) \]
\[ = \{ (1) \} \]
\[ y \]

and pong now follows from the symmetries observed.

The ping-pong argument is forced upon us since (0) and (1) do not provide the tools for a (boolean!) value-preserving transformation of \( x \uparrow y = x \) into \( y \downarrow x = y \). We have to use the semantics of these two equality signs, i.e. we have to apply Leibniz's principle.

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