Problem 10406 from The American Mathematical Monthly, Volume 101, Number 8/October 1994

10406. Proposed by David C. Fisher, University of Colorado, Denver, CO, Karen L. Collins, Wesleyan University, Middleton, CT, and Lucia B. Krompart, Rochester, MI.

Show that a path on an \( m \) by \( n \) square grid which starts at the northwest corner, goes through each point exactly once, and ends at the southeast corner divides the grid into two equal halves: (a) those regions opening north or east; and (b) those regions opening south or west.

(10406) A path meeting the conditions of the problem on a 5 by 8 grid is shown in figure 10406 below.

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  a b b b b a b
  a a a a b a a
  b b b a b b b
  a a a a b a a
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Figure 10406

We confine our attention to paths from NW to SE that do not necessarily go through each point; the off-path points are called free. Directing the paths from NW to SE, I identify (here without proof) the region(s) "a" with those to the left of the path, and the regions "b" with those to its right.

A path, along which turns to the left and turns to the right alternate, is a shortest path; we give a simple example. (in the same 5 by 8 grid).
For the m by n grid with a shortest path we observe

- # points = m * n
- # unit squares = (m-1) * (n-1)
- # edges on shortest path = (m-1) + (n-1)
- # points on shortest path = m + n - 1
- # free points = \( (m \times n) - (m + n - 1) \)
  \= (m-1) \times (n-1)

In short: the number of unit squares equals the number of free points. But we can go further: this equality holds for the separate regions! We can establish in the a-region a 1-1 correspondence between each unit square and its NE corner, and in the b-region between each unit square and its SW corner.

Introducing

\( Fa \) and \( Fb : \) # free points in a-region(s)
and b-region(s) respectively Sa and Sb : # unit squares in a-region(s) and b-region(s) respectively, we can now formulate

Lemma 0 In the case of a shortest path
\[ Fa = S_a \land Fb = S_b \]

The idea of the proof is to start with a given path that visits each point exactly once - i.e. \( Fa = 0 \land Fb = 0 \) - and to transform it in a finite number of moves into a shortest path for which \( Fa = S_a \land Fb = S_b \) holds. Initial and final state are then to be linked via an invariant that is maintained by each move.

For the move we consider that, as said, a path, along which turns to the left and turns to the right alternate, is a shortest path. Consequently, if the path is not of minimal length, it contains at least one pair of successive turns in the same direction. With those turns k edges apart it means that there exists a rectangle - a "bar" - of \( k \times (k+1) \) unit squares such that 3 of its sides are formed by 1, k, 1 edges of the path, e.g. for
k = 5:

\[
\begin{array}{cccccc}
\bullet & x & x & x & x & \bullet \\
\bullet & \cdots \cdots \cdots \cdots \cdots \cdots \bullet \\
\end{array}
\]

We can - and will - always choose a rectangle such that the \( k-1 \) interior points of the 4th side are free. The move shortens the path by 2 edges in the obvious way; in the same example, the above configuration is transformed into

\[
\begin{array}{cccccc}
\bullet & \cdots \cdots \cdots \cdots \cdots \cdots \bullet \\
\bullet & y & y & y & y & y \\
\end{array}
\]

With \( x, y = a, b \) or \( x, y = b, a \), we observe for the transformation by the move:

\[
\begin{align*}
\Delta S_x &= -k \\
\Delta F_x &= -(k-1) \\
\Delta (S_x - F_x) &= -1
\end{align*}
\]

\[
\begin{align*}
\Delta S_y &= +k \\
\Delta F_y &= +k+1 \\
\Delta (S_y - F_y) &= -1
\end{align*}
\]

The above transformation reduces at both sides of the path the difference \( S - F \) by 1, and hence

\[
(S_a - F_a) - (S_b - F_b) = q
\]

if initially true, is an invariant of our total transformation process (which ob-
viously terminates since each move reduces the path by 2 edges).

From Lemma 0, which is applicable in the final state, we conclude $q = 0$. Therefore initially, with $Fa = 0 \wedge Fb = 0$, we have $Sa - Sb = 0$ or $Sa = Sb$. Quod erat demonstrandum.

* * *

It took me several hours to design the move of the previous page; for quite a while I considered 2 unparameterized moves: the current one with $k=1$ and "$\Gamma \rightarrow \Lambda$", which changes the shape of the path without shortening it. The mere fact that the second move complicates the termination argument should have made me reject it much faster.

I liked the problem.

Austin, 17 October 1994

prof. dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712 - 1188
USA