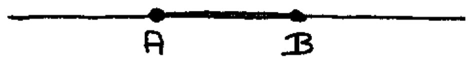
The complete (n+1)-graph in n-dimensional space

We can embed the complete 2-graph with labelled vertices in a 1-dimensional world



or

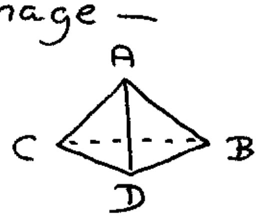
and if the two "ends" of that 1-dimensional world are distinct, the above 2 embeddings are different.

We can embed the complete 3-graph with labelled vertices in a 2-dimensional world

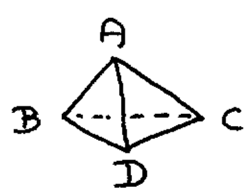
or

and if the two "sides" of the plane are distinct, so are the 2 embeddings.

Similarly —and this will be our last example— the completed 4-graph can be embedded in 2 ways in directed 3-space —i.e. a 3-dimensional world that is distinct from its mirror image—



or



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The first question to be raised and answered is: how do we distinguish, for each number of dimensions, between the 2 embeddings? Moreover we would like to do so in a manner that does not destroy the symmetry between the vertices.

Here the theory of inversions - i.e. (the number of) pairs of elements out of order - gives the answer. For any k, $k \geq 2$, the k! permutations of k elements can be partitioned into 2 classes of equal size such that a single swap transforms any permutation of the one class into a permutation of the other class. For elements A, B, the classes are {AB} and {BA}, for elements A, B, C, the classes are {ABC, BCA, CAB} and {ACB, BAC, CBA}, for elements A, B, C, D, the one class is {ABCD, ACDB, ADBC, BADC, BCAD, BDCA, CABD, CBDA, CDAB, DACB, DBAC, DCBA} and the construction of the other class is left to the reader. I would like to stress that the existence and "shape" of these classes have nothing to do with the labels A, B, C, ... or their alphabetic order: they are intrinsically generated by the process of permuting. The two classes provide the answer to our first question: for any k, we associate the one class with the one embedding, and the other class with the other embedding.

The "oriented" (n+1)-graph has n+1 "faces", one opposite to each vertex. The way to give those faces an orientation in a systematic manner is, for instance, as follows: choose from the representative class for the (n+1)-graph a permutation that starts with the opposite vertex and select the rest of that permutation. So the oriented 3-graph with class {ABC, BCA, CAB} has face BC opposite to A, face CA opposite to B and face AB opposite to C ; the 4-graph corresponding to ABCD has the 4 oriented faces corresponding to BCD, ADC, ABD, and ACB respectively, and any 2 of them contain the subface they share in opposite orientation: e.g. BCD and ADC contain CD and DC respectively, the one having been obtained by truncating a leading AB (from ABCD, as a matter of fact), the other by truncating a leading BA (from BADC). Generalization to more dimensions is left to the reader.

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