The arithmetic and geometric means once more

In the following, \( x, y, c \) are positive.

In \textit{EWD1140}, I used
\[(0) \quad (x+y)^2 = (x-y)^2 + 4 \cdot x \cdot y \]
to argue that
\[(1) \quad x, y := c, x+y-c \quad , \]
which does not change \( x+y \), increases \( x \cdot y \) provided \( c \) lies between the initial values of \( x \) and \( y \).

In \textit{EWD1171}, I used \((0)\) to argue that
\[(2) \quad x, y := c, x \cdot y/c \quad , \]
which does not change \( x \cdot y \), decreases \( x+y \) provided \( c \) lies between the initial values of \( x \) and \( y \).

In both cases the use of \((0)\) came a little bit as a rabbit and the link between the condition on \( c \) and the decrease of the distance between \( x \) and \( y \) remained informal. Last Thursday, when I asked for an expression that contained both \( x+y \) and \( x \cdot y \), my
An Thai Nguyen suggested that we look at 
(c-x)·(c-y), and this expression indeed 
plays a central role in the derivations 
from which all rabbits have been removed.

* * *

We want to change $x, y$ such that 
(i) their sum is not changed, and (ii) 
their product is increased. Any assign-
ment satisfying (i) can be written like 
(1); in order to satisfy (ii) we now 
observe for any $c$

"(1) increases $x·y$"

\[
= \{ \text{program semantics, (1)} \}
\]

\[
x·y < c·(x+y - c)
\]

\[
= \{ \text{algebra} \}
\]

\[
(c-x)·(c-y) < 0
\]

We now consider the change of $x, y$
such that (iii) their product is not 
changed, and (iv) their sum is decreased. 
Any assignment satisfying (iii) can be 
written like (2); in order to satisfy (iv) 
as well, we observe for any positive $c$

"(2) decreases $x+y$"

\[
= \{ \text{program semantics, (2)} \}
\]

\[
c + x·y/c < x+y
\]
\[
\begin{align*}
&= \{ c > 0 \} \\
&\quad c^2 + x \cdot y < c \cdot (x+y) \\
&= \{ \text{algebra} \} \\
&\quad (c-x) \cdot (c-y) < 0
\end{align*}
\]

So, in both cases, the completely forced calculations lead in exactly the same form to the conclusion that \( c \) should lie between the initial values of \( x \) and \( y \). The secret is that Nguyen's expression can be rewritten as
\[
\begin{align*}
&\quad x \cdot y - c \cdot (x+y-c)
\end{align*}
\]
and as
\[
\begin{align*}
&\quad (c^2 + x \cdot y) - (c \cdot x + c \cdot y)
\end{align*}
\]
i.e. the difference of two products with equal sums of their factors, and the difference of two sums with equal products of their addenda. I was surprised.

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