The transitive closure of a wellfounded relation

Transitive closures can be defined in many ways, but today we define the nonreflexive transitive closure $s$ of a relation $r$ as the strongest $s$ satisfying

$(0) \quad [r \lor r; s \equiv s]$

From $(0)$ alone – i.e. not using that $s$ is the strongest – we can derive

$(1) \quad <\forall x:: [x \equiv s; x] \Rightarrow [x \equiv r; x]>$

Proof We observe for an $x$ satisfying

$(2) \quad [x \equiv s; x]$

\[
\begin{align*}
x & \equiv \{ (2) \} \\
&s; x & \equiv \{ (0) \} \\
(r \lor r; s); x & \equiv \{ ; \text{over } \lor \text{ and associative} \} \\
r; x \lor r; s; x & \equiv \{ (2) \} \\
r; x \lor r; x & \equiv \{ \text{pred. calc.} \} \\
r; x
\end{align*}
\]

(End of Proof)
One of the formulations of "r is left-wellfounded" is

\[(3) \quad \langle \forall x :: [x \equiv r; x] \Rightarrow [\neg x]\rangle\]

Thanks to (1), (3) implies

\[(4) \quad \langle \forall x :: [x \equiv s; x] \Rightarrow [\neg x]\rangle\]

in other words: if a relation is left-wellfounded, so is its nonreflexive transitive closure.

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From (0) alone — i.e. not using that s is the strongest — we can derive

\[(4) \quad \langle \forall x :: [x \Rightarrow r; x] \Rightarrow [x \Rightarrow s; x]\rangle\]

**Proof** We observe for any x

\[ [x \Rightarrow s; x] \]

\[\equiv \{ (0) \}

\[ [x \Rightarrow (r \lor r; s); x] \]

\[\iff \{ \text{pred. calc. and monotonicity of ;};\}

\[ [x \Rightarrow r; x] \]

(End of Proof.)

Thanks to Knaster-Tarski, an alternative formulation of "s is left-wellfounded" is

\[(5) \quad \langle \forall x :: [x \Rightarrow s; x] \Rightarrow [\neg x]\rangle\]
From (4) and (5) we derive
\[ \langle \forall x :: [x \Rightarrow r; x] \Rightarrow [\neg x] \rangle \]
in other words: if the nonreflexive transitive closure of a relation is left-wellfounded, so is the relation itself.

**Remark** It is worth noting that the proofs of the crucial implications (1) and (4) use neither wellfoundedness nor the fact that \( s \) is the strongest \( s \) satisfying (0). (End of Remark.)

*   *   *

For left-wellfounded \( r \), (0) determines \( s \) uniquely, i.e. given
\[ (6) \quad [r \lor r; s \equiv s] \]
\[ (7) \quad [r \lor r; t \equiv t] \]
\[ (8) \quad \langle \forall x :: [x \Rightarrow r; x] \Rightarrow [\neg x] \rangle \]

we have to prove \([s \equiv t]\)

**Proof** For reasons of symmetry, it suffices to prove \([t \Rightarrow s]\). We observe
\[ [t \Rightarrow s] \]
\[ \equiv \quad \{ \text{pred. calc.} \} \]
\[ [\neg (t \land \neg s)] \]
\[ \equiv \quad \{ (8) \text{ with } x := t \land \neg s \} \]
\[ t \land s \Rightarrow r; (t \land s) \]
\[ = \{ \text{shunting and (6)} \} \]
\[ t \Rightarrow r \lor s; s \lor r; (t \land s) \]
\[ = \{ ; \text{ over } \lor \text{ and pred.calc.} \} \]
\[ t \Rightarrow r \lor r; (t \lor s) \]
\[ = \{ \text{monotonicity of } ; \} \]
\[ t \Rightarrow r \lor r; t \]
\[ = \{ (7) \} \]
true
(End of Proof.)

The above is a considerable streamlining of Avg 88/EWD 1079 dd 28 April 1990 (which made no use of the relational calculus).

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prof. dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712 - 1188
USA