The formula for \( \sin(\alpha + \beta) \)

We consider a triangle with sides a, b, c and opposite angles \(\alpha, \beta, \gamma\) respectively:

We have added the altitude CF; the additional annotation follows from the definitions of the sine and cosine functions. We observe

\[
\begin{align*}
\text{true} & \quad \{ \text{the two annotations for } CF \} \\
& \quad \{ \text{algebra} \} \\
& \quad \{ \text{symmetry} \} \\
& \quad a : b : c = \sin \alpha : \sin \beta : \sin \gamma \quad (\star)
\end{align*}
\]

Next we observe

true

\[
\begin{align*}
& \quad \{ \text{annotations for } BF \text{ and } FA \} \\
\end{align*}
\]
\[ c = a \cdot \cos \beta + b \cdot \cos \alpha \]  \( (*) \)
\[ \equiv \{ (**) \} \]
\[ \sin \gamma = \sin \alpha \cdot \cos \beta + \sin \beta \cdot \cos \alpha \]
\[ \equiv \{ \alpha + \beta + \gamma = \pi \} \]
\[ \sin (\alpha + \beta) = \sin \alpha \cdot \cos \beta + \sin \beta \cdot \cos \alpha \]  \( (**) \)

and so we have proved the addition formula \( (**) \) for the sine function for \( 0 \leq \alpha, 0 \leq \beta \) and \( \alpha + \beta \leq \pi \). (Note that \( \gamma \) does not need to lie between \( A \) and \( B \) for \( (*) \) to be valid.)

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