The ladder theorem

Let $p$ and $q$ be two column vectors of the same (finite) length. Let $[p \geq q]$ denote that, in each row, the $p$-element is at least the $q$-element, i.e.

$$[p \geq q] \equiv (\forall i : p.i \geq q.i)$$

Let $\text{sort.w}$ denote the result of sorting column vector $w$ in ascending order. Then the ladder theorem states

$$\text{(0)} \quad [p \geq q] \Rightarrow [\text{sort.p} \geq \text{sort.q}]$$

Proof. We observe for any $x$ and positive $n$

- the $n$th element of $\text{sort.p}$ is $\leq x$,
- $\{\text{sort.p is in ascending order}\}$
- $\text{sort.p}$ contains at least $n$ elements $\leq x$
- $\{\text{p is a permutation of sort.p}\}$
- $p$ contains at least $n$ elements $\leq x$
- $\Rightarrow \{[p \geq q], \text{the antecedent of (0)}\}$
- $q$ contains at least $n$ elements $\leq x$
- $\{q \text{ is a permutation of sort.q}\}$
- $\text{sort.q}$ contains at least $n$ elements $\leq x$
- $\{\text{sort.q is in ascending order}\}$
- the $n$th element of $\text{sort.q}$ is $\leq x$,

and since the above calculation derives
that, as a consequence of \([p \geq q]\), its first line implies its last line for any \(x\), it captures by "indirect order" the elementwise demonstration of \([\text{sort}.p \geq \text{sort}.q]\).

(End of Proof.)

The ladder theorem is well-known. It tells us that if, in a matrix with sorted rows, we sort all the columns, the rows remain sorted. The above proof of the ladder theorem has been recorded (i) because there are such messy proofs of it -I saw one the other month-, (ii) because I don't succeed in viewing it as "intuitively obvious", and (iii) because the above proof is so nicely disentangled: sorting is permuting and making ascending, and please note the separate appeals to these two properties.

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