An unavoidable case analysis

Archiving old manuscripts, I found at the end of EWD766 "An educational stupidity" of more than 20 years ago the following exercise:

"Prove that none of the decimal numbers 1001, 1001001, 1001001001, 1001001001001, ... is prime." [It is not clear why I underlined "none". EWD]

Here is a proof. Denoting by \( k^{*\text{string}^*} \) the concatenation of \( k \) copies of the digit string enclosed, we deal with the decimal numbers \( k^{*001^*} \) for \( k \geq 2 \)

(i) If \( k \mod 3 = 0 \), the number is, according to the traditional 3-test, divisible by 3 because the sum of its decimal digits, which equals \( k \), is divisible by 3.

(ii) If \( k \mod 3 \neq 0 \), we see by generalizing the traditional 9-test to the \( k^{*9^*} \) test, that the number reduced modulo \( k^{*9^*} \) equals \( k^{*1^*} \). Hence the number is divisible by \( k^{*1^*} \) (because \( k^{*9^*} \) is). (Note that this observation does not exclude primality for \( k=1 \)).

In argument (i), the crux is the validity
of the 3-test, which is valid whenever base \( \mod 3 = 1 \), a condition that base 10 (= ten) satisfies. The conclusion has nothing to do with the accident that the length of the repeated string happens to be 3: for \( k \mod 3 = 0 \), also \( k*00001* \) is divisible by 3.

In argument (ii) we generalized the 9-test because the problem was about decimal numbers, but for any base \( B \) \( (B \geq 2) \) there is a \( (B-1) \)-test, and \( k*1* \) divides \( k*(B-1)* \). The conclusion that the number reduced modulo \( k*(B-1)* \) yields \( k*1* \), however, depends on the fact that \( k \) has no factor in common with the length of the iterated string. Argument (ii) is base independent: interpreting \( *k001* \) and \( *k1* \) in binary, we find for instance that for \( k=4 \) and \( k=5 \), 585 is divisible by 15 and 4681 by 31.

Argument (i) relies on a relation between \( k \) and the base of the number system, (ii) on a relation between \( k \) and the length of the iterated string.

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