# Geometric Spaces and Operations

Mathematical underpinnings of computer graphics

- Hierarchy of geometric spaces
  - Vector spaces
  - Affine spaces
  - Euclidean spaces
  - Cartesian spaces
  - Projective spaces
- Affine geometry and transformations
- Projective transformations and perspective
- Matrix formulations of transformations

Formally, a space is defined by

- A set of objects

- Operations on the objects
- Axioms defining invariant properties

# Vector Spaces

*Definition:*

- Set of *vectors* $\mathcal{V}$
- Operations on $\vec{u},\ \vec{v} \in \mathcal{V}$:
  - *Addition:* $\vec{u} + \vec{v} \in \mathcal{V}$
  - *Scalar Multiplication:* $\alpha\vec{u} \in \mathcal{V}$ where $\alpha \in$ some field $\mathcal{F}$
- *Axioms*
  - *Unique zero element:* $0 + \vec{u} = \vec{u}$
  - *Field unit element:* $1\vec{u} = \vec{u}$
  - *Addition commutative:* $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
  - *Addition associative:* $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
  - *Distributive scalar multiplication:* $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$
- Additional definitions
  - Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$.
  - Then $\mathcal{B}$ *spans* $\mathcal{V}$ iff any $\vec{v} \in \mathcal{V}$ can be written as $\vec{v} = \sum_{i=1}^{n} \alpha_i \vec{v}_i$.
  - $\sum_{i=1}^{n} \alpha_i \vec{v}_i$ is called a *linear combination* of the vectors in $\mathcal{B}$.
  - $\mathcal{B}$ is called a *basis* of $\mathcal{V}$ if it is a minimal spanning set.
  - All bases of $\mathcal{V}$ contain the same number of vectors.

  - The number of vectors in any basis of $\mathcal{V}$ is called the *dimension* of $\mathcal{V}$.
- Comments:
  - We are interested in 2 and 3 dimensional spaces.
  - No definition of distance (size) exists yet.
  - Angles and points have not been defined.

# Affine Spaces

*Definition:*

- A set of vectors $\mathcal{V}$ and a set of *points* $\mathcal{P}$
- $\mathcal{V}$ is a vector space.
- Point-vector sum: $P + \vec{v} = Q$ with $P,\ Q \in \mathcal{P}$ and $\vec{v} \in \mathcal{V}$
- Additional definitions:
  - A *frame* $F = (\mathcal{B}, \mathcal{O})$ where $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is a basis of $\mathcal{V}$ and the point $\mathcal{O}$ is called the *origin* of the frame.
  - The dimension of $F$ is the same as the dimension of $\mathcal{V}$.
- Comments:
  - Still no distances or angles
  - Closer to what we want for graphics
  - The space has no distinguished origin

# Euclidean Spaces

*Definition:*

- A *metric space* is any space with a *distance metric* $d(P, Q)$ defined on its elements.
- Distance metric axioms:
  - $d(P, Q) \geq 0$
  - $d(P, Q) = 0$ iff $P = Q$
  - $d(P, Q) = d(Q, P)$
  - $d(P, Q) \leq d(P, R) + d(R, Q)$ (triangle inequality)
- *Euclidean* distance metric:

$$d^2(P, Q) = (P - Q) \cdot (P - Q)$$

- Comments:
  - Euclidean metric based on dot product
  - Dot product defined on vectors
  - Distance metric defined on points
  - Distance is a property of the space, not a frame

- Dot product axioms:
  - $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
  - $\alpha(\vec{u} \cdot \vec{v}) = (\alpha\vec{u}) \cdot \vec{v} = \vec{u} \cdot (\alpha\vec{v})$
  - $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- Additional definitions:
  - The *norm* of a vector $\vec{u}$ is given by $|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$.
  - Angles are defined by their cosines: $\cos(\angle\vec{u}\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$
  - Orthogonal vectors: $\vec{u} \cdot \vec{v} = 0 \rightarrow \vec{u}\perp\vec{v}$

# Cartesian Spaces

*Definition:*

- A frame $(\vec{i}, \vec{j}, \vec{k}, \mathcal{O})$ is *orthonormal* iff
  - $\vec{i}, \vec{j}$, and $\vec{k}$ are *orthogonal*, i.e. $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$ and
  - $\vec{i}, \vec{j}$, and $\vec{k}$ are *normal*, i.e. $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$
- Additional definitions:
  - The *standard frame* $F_s = (\vec{i}, \vec{j}, \vec{k}, \mathcal{O})$
  - Points can be distinguished from vectors using an extra coordinate
    * 0 for vectors: $\vec{v} = (v_x, v_y, v_z, 0)$ means $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$
    * 1 for points: $P = (p_x, p_y, p_z, 1)$ means $P = p_x\vec{i} + p_y\vec{j} + p_z\vec{k} + \mathcal{O}$
- Comments
  - Coordinates have no meaning without an associated frame
  - There will be other ways to look at the extra coordinate
  - Sometimes we are sloppy and omit the extra coordinate
  - Assume standard frame unless specified otherwise
  - Points and vectors are different
  - Points and vectors have different operations
  - Points and vectors transform differently

# Projective Space

*Homogeneous Coordinates:*

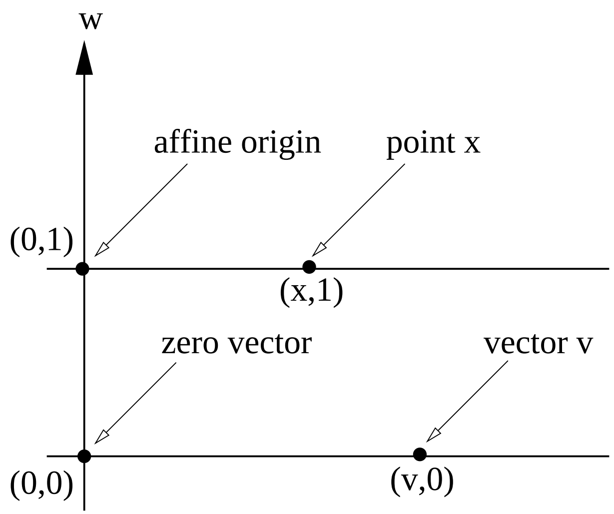
- Reminders
- Notation
- Imbedding

*Projective Space:*

- Division by the homogeneous coordinate
- Equivalence of affine points and homogeneous points
- Relationship with perspective
- More generally: rational splines

# Homogeneous Coordinates

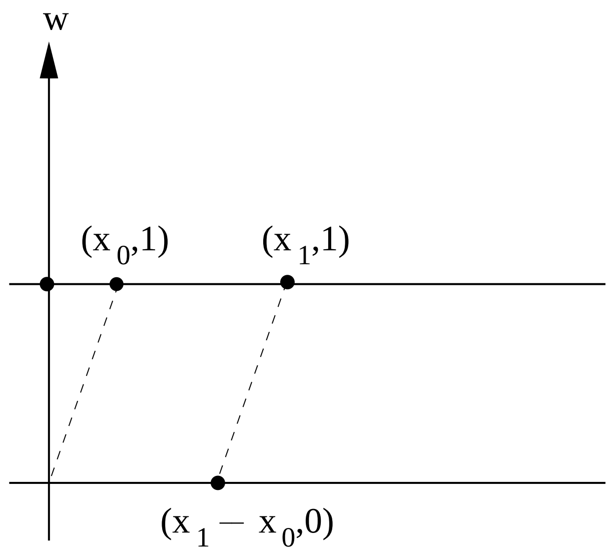
- Reminder about Spaces
  - Affine Space = Vector Space + Points
- Homogeneous notation
  - vector: $(x, y, z, 0)$
  - point: $(x, y, z, 1)$
- Imbedding of vectors and points in space of one higher dimension



# Vector and Affine Algebra

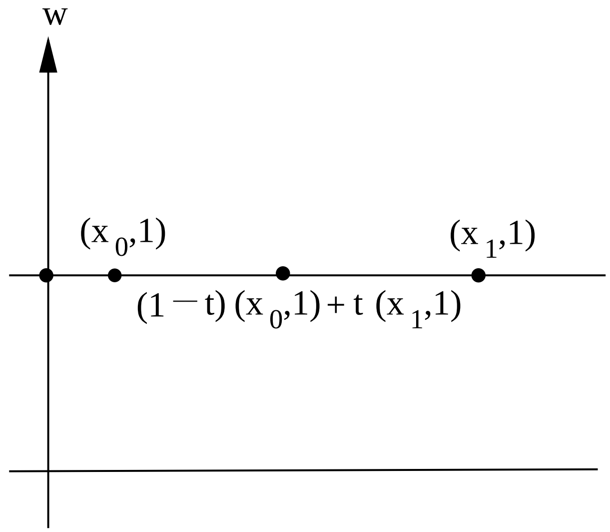
- Difference of points

$$(x_1, 1) - (x_0, 1) = (x_1 - x_0, 0)$$



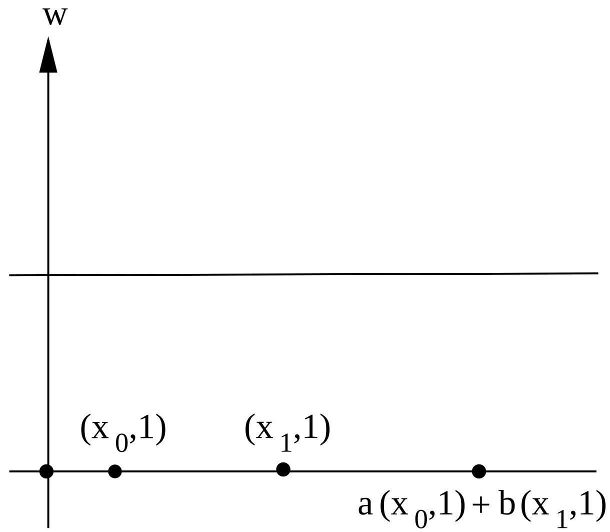
- Affine combination of points

$$(1-t)(x_1, 1) + t(x_0, 1) = ((1-t)x_1 + tx_0, 1)$$



- Linear combinations of vectors

$$a(v_0, 0) + b(v_1, 0) = (av_0 + bv_1, 0)$$

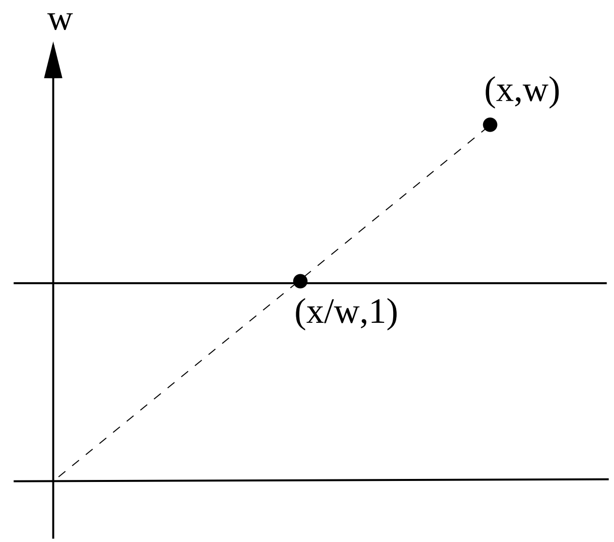


# Homogeneous Coordinates

- Homogeneous coordinates represent $n$-space as a subspace of $n + 1$ space
- For instance, homogeneous 4-space embeds ordinary 3-space as the $w = 1$ hyperplane
- Thus, we can obtain the 3-d image of any homogeneous point $(wx, wy, wz, w)$, $w \neq 0$ as $(x, y, z, 1) = (wx/w, wy/w, wz/w, w/w)$, that is, by dividing all coordinates by $w$.
- Lines in homogeneous space which intersect the $w = 1$ hyperplane project to 3-space points.
- Notice that this is just a perspective projection from 4-d homogeneous space to 3-space, instead of dividing by $z$, we are dividing by $w$.

# Projective Space

- Divide through by $w$
  $(x, w) \longrightarrow (\frac{x}{w}, 1)$
- All homogeneous points of the form $\alpha(x, 1), \alpha > 0$ are equivalent
- *Projects* homogeneous points *centrally* onto the affine plane



*Relationship to Perspective:*

- In rendering, the $w$ values we generate are proportional to $z$
  - Equivalence corresponds to *perspective projection*

*More Generally:* Rational splines

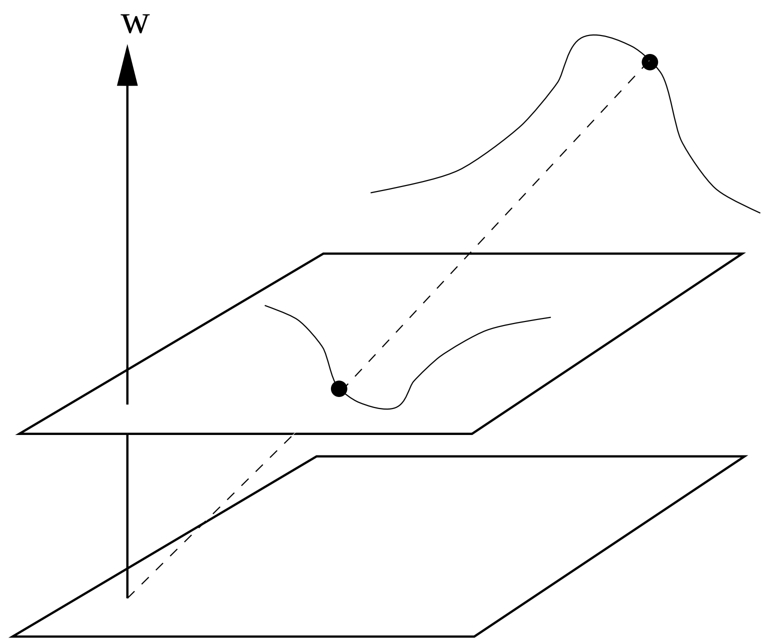
- Homogeneous spline curve
  - Spline curve of the form

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{bmatrix} = \sum_{i=0}^{n} \begin{bmatrix} x_i \\ y_i \\ z_i \\ w_i \end{bmatrix} B_i^d(t) = \begin{bmatrix} \sum_{i=0}^{n} x_i B_i^d(t) \\ \sum_{i=0}^{n} y_i B_i^d(t) \\ \sum_{i=0}^{n} z_i B_i^d(t) \\ \sum_{i=0}^{n} w_i B_i^d(t) \end{bmatrix}$$

- Rational spline curve
  - Affine projective spline curve

$$\begin{bmatrix} \bar{x}(t) \\ \bar{y}(t) \\ \bar{z}(t) \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n} x_i B_i^d(t) / \sum_{i=0}^{n} w_i B_i^d(t) \\ \sum_{i=0}^{n} y_i B_i^d(t) / \sum_{i=0}^{n} w_i B_i^d(t) \\ \sum_{i=0}^{n} z_i B_i^d(t) / \sum_{i=0}^{n} w_i B_i^d(t) \end{bmatrix}$$

$$= \frac{1}{\sum_{i=0}^{n} w_i B_i^d(t)} \sum_{i=0}^{n} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} B_i^d(t)$$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# Linear Transformations

Vector space $\mathcal{V}$

- Linear combinations of vectors in $\mathcal{V}$ are in $\mathcal{V}$
- For $\vec{u}, \vec{v} \in \mathcal{V}$
  - $\vec{u} + \vec{v} \in \mathcal{V}$
  - $\alpha\vec{u} \in \mathcal{V}$ for any scalar $\alpha$
  - In general, $\sum_i \alpha_i \vec{u}_i \in \mathcal{V}$ for any scalars $\alpha_i$
- Linear transformations
  - Let $\mathbf{T} : \mathcal{V}_0 \mapsto \mathcal{V}_1$, where $\mathcal{V}_0$ and $\mathcal{V}_1$ are vector spaces
  - Then $\mathbf{T}$ is *linear* iff
    * $\mathbf{T}(\vec{u} + \vec{v}) = \mathbf{T}(\vec{u}) + \mathbf{T}(\vec{v})$
    * $\mathbf{T}(\alpha\vec{u}) = \alpha\mathbf{T}(\vec{u})$
    * In general, $\mathbf{T}\left(\sum_i \alpha_i \vec{u}_i\right) = \sum_i \alpha_i \mathbf{T}\left(\vec{u}_i\right)$

# Affine Transformations

Affine space $\mathcal{A} = (\mathcal{V}, \mathcal{P})$

- For $\vec{u} \in \mathcal{V}$ and $P \in \mathcal{P}$

$$P + \vec{u} \in \mathcal{P}$$

- Define *point subtraction*:
  - For $P, Q \in \mathcal{P}$ and $\vec{u} \in \mathcal{V}$, if $P + \vec{u} = Q$, then $Q - P \equiv \vec{u}$
  - So in general we have $\sum_i \alpha_i P_i$ is a *vector* iff $\sum_i \alpha_i = 0$
- Define *point blending*:
  - For $P, P_1, P_2 \in \mathcal{P}$ and scalar $\alpha$, if $P = P_1 + \alpha\,(P_2 - P_1)$ then $P \equiv (1 - \alpha)\,P_1 + \alpha P_2$
  - This can also be written $P \equiv \alpha_1 P_1 + \alpha_2 P_2$ where $\alpha_1 + \alpha_2 = 1$
  - So in general we have $\sum_i \alpha_i P_i$ is a *point* iff $\sum_i \alpha_i = 1$
- Geometrically, we have $\frac{|P-P_0|}{|P-P_1|} = \frac{d_1}{d_2}$ or $P = \frac{d_1 P_1 + d_2 P_2}{d_1 + d_2}$
- Vectors can always be combined linearly $\sum_i \alpha_i \vec{u}_i$
- Points can be combined linearly $\sum_i \alpha_i P_i$ iff
  - The coefficients sum to 1, giving a point (“affine combination”)
  - The coefficients sum to 0, giving a vector (“vector combination”)

- Example affine combination:

$$P(t) = P_0 + t(P_1 - P_0) = (1 - t)P_0 + tP_1$$

- This says any point on the line is an affine combination of the line segment's endpoints.

- Affine transformations
  - Let $\mathbf{T} : \mathcal{A}_0 \mapsto \mathcal{A}_1$ where $\mathcal{A}_0$ and $\mathcal{A}_1$ are affine spaces
  - $\mathbf{T}$ is said to be an *affine transformation* iff
    * $\mathbf{T}$ maps vectors to vectors and points to points
    * $\mathbf{T}$ is a linear transformation on the vectors
    * $\mathbf{T}(P + \vec{u}) = \mathbf{T}(P) + \mathbf{T}(\vec{u})$
  - Properties of affine transformations
    * $\mathbf{T}$ preserves affine combinations:

$$\mathbf{T}(\alpha_0 P_0 + \cdots + \alpha_n P_n) = \alpha_0 \mathbf{T}(P_0) + \cdots + \alpha_n \mathbf{T}(P_n)$$

where $\sum_i \alpha_i = 0$ or $\sum_i \alpha_i = 1$

    * $\mathbf{T}$ maps lines to lines:

$$\mathbf{T}((1 - t)P_0 + tP_1) = (1 - t)\mathbf{T}(P_0) + t\mathbf{T}(P_1)$$

  * $\mathbf{T}$ is affine iff it preserves ratios of distance along a line:

$$P = \frac{d_0 P_0 + d_1 P_1}{d_0 + d_1} \Rightarrow \mathbf{T}(P) = \frac{d_0 \mathbf{T}(P_0) + d_1 \mathbf{T}(P_1)}{d_0 + d_1}$$

  * $\mathbf{T}$ maps parallel lines to parallel lines (can you prove this?)
- Example affine transformations
  * Rigid body motions (translations, rotations)
  * Scales, reflections
  * Shears

# Matrix Representation of Transformations

- Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be affine spaces.
  Let $\mathbf{T} : \mathcal{A}_0 \mapsto \mathcal{A}_1$ be an affine transformation.
  Let $F_0 = (\vec{i}_0, \vec{j}_0, \mathcal{O}_0)$ be a frame for $\mathcal{A}_0$.
  Let $F_1 = (\vec{i}_1, \vec{j}_1, \mathcal{O}_1)$ be a frame for $\mathcal{A}_1$.
- Let $P = x\vec{i}_0 + y\vec{j}_0 + \mathcal{O}_0$ be a point in $\mathcal{A}_0$.
  The *coordinates* of $P$ relative to $\mathcal{A}_0$ are $(x, y, 1)$.

  This can also be represented in vector form as $P = \left[\vec{i}_0\ \vec{j}_0\ \mathcal{O}_0\right] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

- What are the coordinates $(x', y', 1)$ of $\mathbf{T}(P)$ relative to $F_1$?
  - An affine transformation is characterized by the image of a frame in the domain.

$$\begin{aligned}\mathbf{T}(P) &= \mathbf{T}(x\vec{i}_0 + y\vec{j}_0 + \mathcal{O}_0) \\ &= x\mathbf{T}(\vec{i}_0) + y\mathbf{T}(\vec{j}_0) + \mathbf{T}(\mathcal{O}_0)\end{aligned}$$

  - $\mathbf{T}(\vec{i}_0)$ must be a linear combination of $\vec{i}_1$ and $\vec{j}_1$, say $\mathbf{T}(\vec{i}_0) = t_{1,1}\vec{i}_1 + t_{2,1}\vec{j}_1$.
  - Likewise $\mathbf{T}(\vec{j}_0)$ must be a linear combination of $\vec{i}_1$ and $\vec{j}_1$, say $\mathbf{T}(\vec{j}_0) = t_{1,2}\vec{i}_1 + t_{2,2}\vec{j}_1$.
  - Finally $\mathbf{T}(\mathcal{O}_0)$ must be an affine combination of $\vec{i}_1$, $\vec{j}_1$, and $\mathcal{O}_1$, say $\mathbf{T}(\mathcal{O}_0) = t_{1,3}\vec{i}_1 + t_{2,3}\vec{j}_1 + \mathcal{O}_1$.

– Then by substitution we get

$$\begin{aligned}\mathbf{T}(P) &= x(t_{1,1}\vec{i}_1 + t_{2,1}\vec{j}_1) + y(t_{1,2}\vec{i}_1 + t_{2,2}\vec{j}_1) + t_{1,3}\vec{i}_1 + t_{2,3}\vec{j}_1 + \mathcal{O}_1 \\ &= \left[t_{1,1}\vec{i}_1 + t_{2,1}\vec{j}_1 \;\; t_{1,2}\vec{i}_1 + t_{2,2}\vec{j}_1 \;\; t_{1,3}\vec{i}_1 + t_{2,3}\vec{j}_1 + \mathcal{O}_1\right] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ &= \left[\vec{i}_1 \; \vec{j}_1 \; \mathcal{O}_1\right] \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\end{aligned}$$

Using $\mathbf{M}_T$ to denote the matrix, we see that $F_0 = F_1\mathbf{M}_T$

- Let $\mathbf{T}(P) = P' = x'\vec{i}_1 + y'\vec{j}_1 + \mathcal{O}_1$

In vector form this is

$$\begin{aligned} P' &= \left[\vec{i}_1\ \vec{j}_1\ \mathcal{O}_1\right] \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \\ &= \left[\vec{i}_1\ \vec{j}_1\ \mathcal{O}_1\right] \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \end{aligned}$$

So we see that

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

We can write this in shorthand – $\mathbf{p}' = \mathbf{M}_T\mathbf{p}$

- $\mathbf{M}_T$ is the *matrix representation* of **T**
  - The first column of $\mathbf{M}_T$ represents $\mathbf{T}(\vec{i}_0)$
  - The second column of $\mathbf{M}_T$ represents $\mathbf{T}(\vec{j}_0)$
  - The third column of $\mathbf{M}_T$ represents $\mathbf{T}(\mathcal{O}_0)$

- *Translation*
  - Points are transformed as $\left[x'\ y'\ 1\right]^T = [x\ y\ 1]^T + [\Delta x\ \Delta y\ 0]^T$.
  - Vectors don't change.
  - Thus translation is affine but not linear.
    If it were linear, we would have $\mathbf{T}(P+Q) = \mathbf{T}(P) + \mathbf{T}(Q)$, but point addition is undefined.
  - Translation can be applied to sums of vectors and vector-point sums.
  - Matrix formulation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + \Delta x \\ y + \Delta y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

  - Shorthand for the above matrix: $T(\Delta x, \Delta y)$

- *Scale*
  - Linear transform — applies equally to points and vectors
  - Points transform as $[x' \; y' \; 1]^T = [xS_x \; yS_y \; 1]^T$.
  - Vectors transform as $[x' \; y' \; 0]^T = [xS_x \; yS_y \; 0]^T$.
  - Matrix formulation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} xS_x \\ yS_y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} xS_x \\ yS_y \\ 0 \end{bmatrix}$$

  - Shorthand for the above matrix: $S(S_x, S_y)$
  - Note that this is *origin sensitive*.
  - How do you do reflections?

- *Rotate*
  - Linear transform — applies equally to points and vectors
  - Points transform as
    $\left[x'\ y'\ 1\right]^T = [x\cos(\theta) - y\sin(\theta)\ \ x\sin(\theta) + y\cos(\theta)\ \ 1]^T$.
  - Vectors transform as
    $\left[x'\ y'\ 0\right]^T = [x\cos(\theta) - y\sin(\theta)\ \ x\sin(\theta) + y\cos(\theta)\ \ 0]^T$.
  - Matrix formulation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \\ 0 \end{bmatrix}$$

  - Shorthand for the above matrix: $R(\theta)$
  - Note that this is *origin sensitive*.

- *Shear*
  - Linear transform — applies equally to points and vectors
  - Points transform as $\left[x' \; y' \; 1\right]^T = [x + \alpha y, \; y + \beta x, \; 1]^T$.
  - Vectors transform as $\left[x' \; y' \; 0\right]^T = [x + \alpha y, \; y + \beta x, \; 0]^T$.
  - Matrix formulation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + \alpha y \\ y + \beta x \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x + \alpha y \\ y + \beta x \\ 0 \end{bmatrix}$$

  - Shorthand for the above matrix: $Sh(\alpha, \beta)$

- Composition of Transformations
  - Now we have some basic transformations, how do we create and represent arbitrary affine transformations?
  - We can derive an arbitrary affine transform as a sequence of basic transformations, then compose the transformations
  - Example — scaling about an arbitrary point $[x_c\ y_c\ 1]^T$
    1. Translate $[x_c\ y_c\ 1]^T$ to $[0\ 0\ 1]$ $(T(-x_c, -y_c))$
    2. Scale $[x'\ y'\ 1]^T = S(S_x, S_y)\,[x\ y\ 1]^T$
    3. Translate $[0\ 0\ 1]^T$ back to $[x_c\ y_c\ 1]$ $(T(x_c, y_c))$
  - The sequence of transformation steps is
    $T(-x_c, -y_c) \circ S(S_x, S_y) \circ T(x_c, y_c)$

– In matrix form this is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} S_x & 0 & x_c(1 - S_x) \\ 0 & S_y & y_c(1 - S_y) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

– Note that the matrices are arranged from *right to left* in the order of the steps.

– The order is important (why)?

- Three Dimensional Transformations
  - A point is $\mathbf{p} = [x\ y\ z\ 1]$, a vector $\vec{v} = [x\ y\ z\ 0]$
  - Translation:

$$T(\Delta x, \Delta y, \Delta z) = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - Scale:

$$S(S_x, S_y, S_z) = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - Rotation:

$$R_z(\Theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Projections and Projective Transformations

*Perspective Projection*

- Identify all points with a line through the eyepoint.
- Slide lines with viewing plane, take intersection point as projection.
- This is *not* an affine transformation, but a *projective transformation.*

*Projective Transformations:*

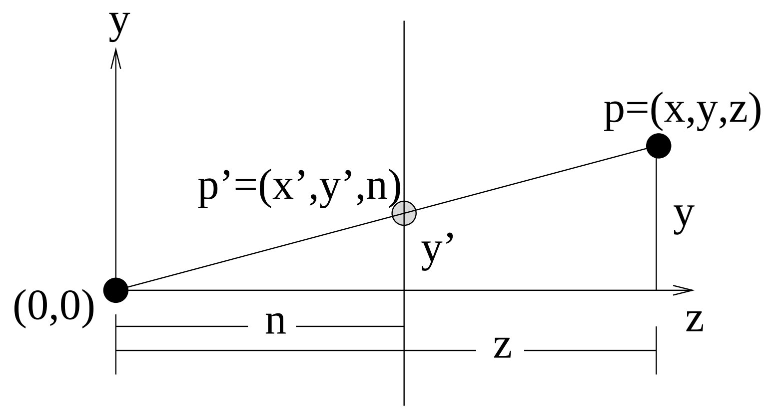
- Angles are not preserved.
- Distances are not preserved.
- Ratios of distances are not preserved.
- Affine combinations are not preserved.
- Straight lines are mapped to straight lines.
- Incidence relationships are preserved in a general way.
- *Cross ratios* are preserved.

# Comparisons

| *Affine Transformations* | *Projective Transformations* |
|---|---|
| Image of 2 points on a line determine image of line | Image of 3 points on a line determine image of line |
| Image of 3 points on a plane determine image of plane | Image of 4 points on a plane determine image of plane |
| In dimension $n$ space, image of $n+1$ points/vectors defines affine map. | In dimension $n$ space, image of $n+2$ points/vectors defines projective map. |
| Vectors map to vectors | Vectors can map to vectors or points |
| Points map to points | Points can map to vectors or points |
| Can represent with matrix multiply | Can represent with matrix multiply and normalization. |

## Perspective Map

- Given a point $P$, we want to find its projection $P'$.




Projection plane, z = n

- Similar triangles: $P' = (xn/z, n)$
- In 3D, $(x', y', z') \mapsto (xn/z, yn/z, n)$
- Have identified all points on a line through the origin with a point in the projection plane.
- Thus, $(x, y, z) \equiv (kx, ky, kz)$, $k \neq 0$.
- These are known as homogeneous coordinates.
- If we have solids, or colored lines, then we need to know "which one is in front."
- This map loses all $z$ information, so it is inadequate.

# Why Map Z

- 3D $\mapsto$ 2D projections map all $z$ to same value.
- Need $z$ to determine occlusion, so a 3D to 2D projective transformation doesn't work.
- Further, we want 3D lines to map to 3D lines (this is useful in hidden surface removal).
- The mapping $(x, y, z, 1) \mapsto (xn/z, yn/z, n, 1)$ maps lines to lines, but loses all depth information.
- We could use

$$(x, y, z, 1) \mapsto (xn/z, yn/z, z, 1)$$

Thus, if we map the endpoints of a line segment, these end points will have the same relative depths after this mapping.

BUT: It fails to map lines to lines

- The map

$$(x, y, z, 1) \mapsto \left(\frac{xn}{z}, \frac{yn}{z}, \frac{zf + zn - 2fn}{z(f - n)}, 1\right)$$

*does* map lines to lines, *and* it preserves depth information.

# Mapping Z

- It's clear how $x$ and $y$ map. How about $z$?

$$z \mapsto \frac{zf + zn - 2fn}{z(f-n)} = P(z)$$

- We know $P(f) = 1$ and $P(n) = -1$. What maps to 0?

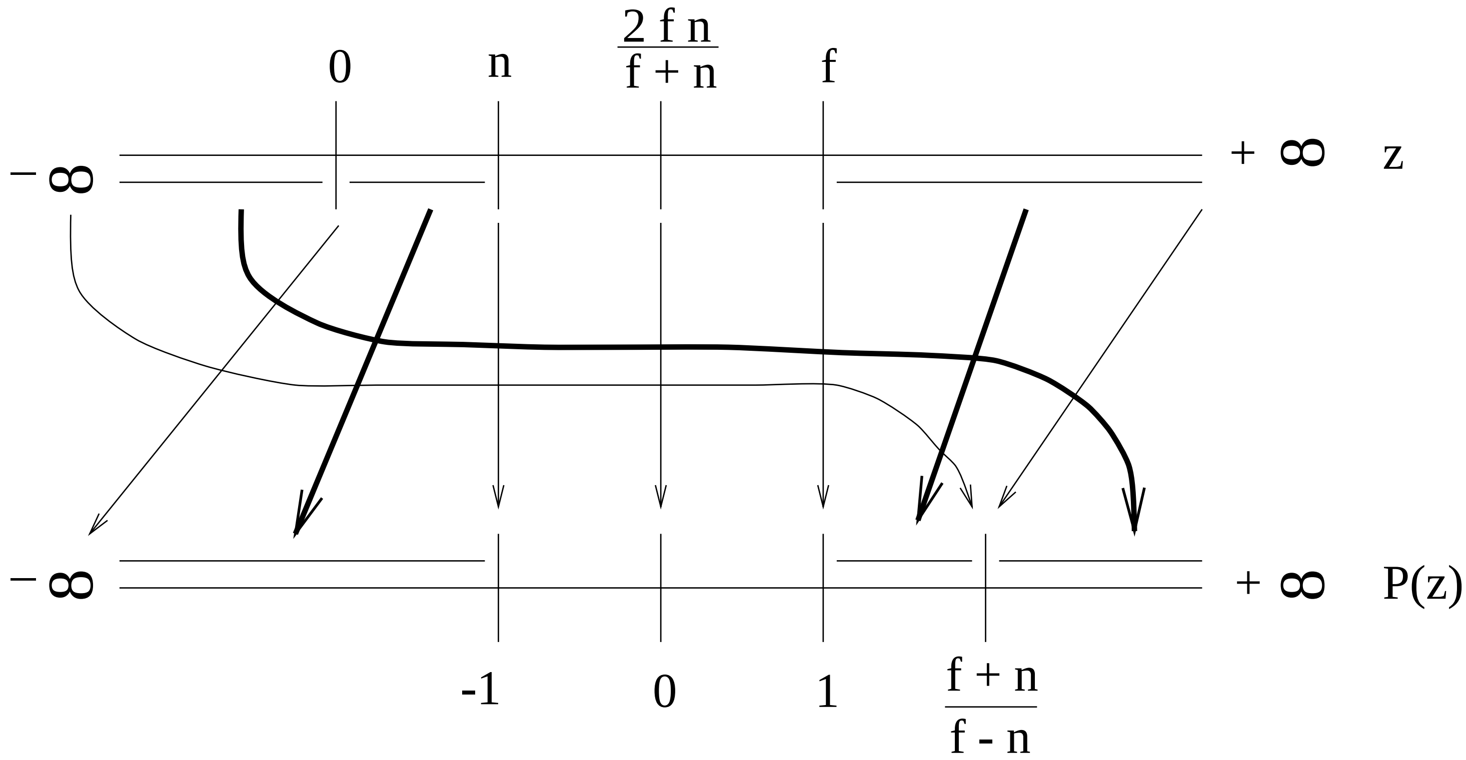
$$\begin{aligned} & & P(z) &= 0 \\ \Rightarrow & & \frac{zf + zn - 2fn}{z(f-n)} &= 0 \\ \Rightarrow & & z &= \frac{2fn}{f+n} \end{aligned}$$

Note that $f^2 + 2f > 2fn/(f+n) > fn + n^2$ so

$$f > \frac{2fn}{f+n} > n$$

- What happens as map $z$ to 0 or to infinity?

$$\begin{aligned}
\lim_{z\to 0^+} P(z) &= \frac{-2fn}{z(f-n)} \\
&= -\infty \\
\lim_{z\to 0^-} P(z) &= \frac{-2fn}{z(f-n)} \\
&= +\infty \\
\lim_{z\to +\infty} P(z) &= \frac{z(f+n)}{z(f-n)} \\
&= \frac{f+n}{f-n} \\
\lim_{z\to -\infty} P(z) &= \frac{z(f+n)}{z(f-n)} \\
&= \frac{f+n}{f-n}
\end{aligned}$$


0
n
2 f n / f + n
f
− ∞
+ ∞
z
− ∞
+ ∞
P(z)
-1
0
1
f + n / f - n

- What happens if we vary $f$ and $n$?

$$\lim_{f \to n} P(z) = \frac{z(f+n) - 2fn}{z(f-n)}$$
$$= \frac{(2zn - 2n^2)}{z \cdot 0}$$

which is not surprising, since we're trying to map a single point to a line segment.

$$\lim_{f \to \infty} P(z) = \frac{zf - 2fn}{zf}$$
$$= \frac{z - 2n}{z}$$

- But note that this means we are mapping an infinite region to [0,1] and we will effectively get a far plane due to floating point precision,

$$\lim_{n \to 0} P(z) = \frac{zf}{zf}$$
$$= 1$$

i.e., the entire map becomes constant (again, we are mapping a point to an interval).

- Consider what happens as $f$ and $n$ move away from each other.
  - We are interested in the size of the regions $[n, 2fn/(f+n)]$ and $[2fn/(f+n), f]$ .
  - When $f$ is large compared to $n$, we have

$$\frac{2fn}{f+n} \doteq 2n$$

So

$$\frac{2fn}{f+n} - n \doteq n$$

and

$$f - \frac{2fn}{f+n} \doteq f - 2n$$

But both intervals are mapped to a regions of size 1.

  - Thus, as we move the clipping planes away from one another, the far interval is compressed more than the near one. With floating point arithmetic, this means we'll lose precision.
  - In the extreme case, think about what happens as we move $f$ to infinity: we compress an infinite region to an finite one.
  - Therefore, we try to place our clipping planes as close to one another as we can.

# Clipping in Homogeneous Space

*Projection: linear transformations then normalize*

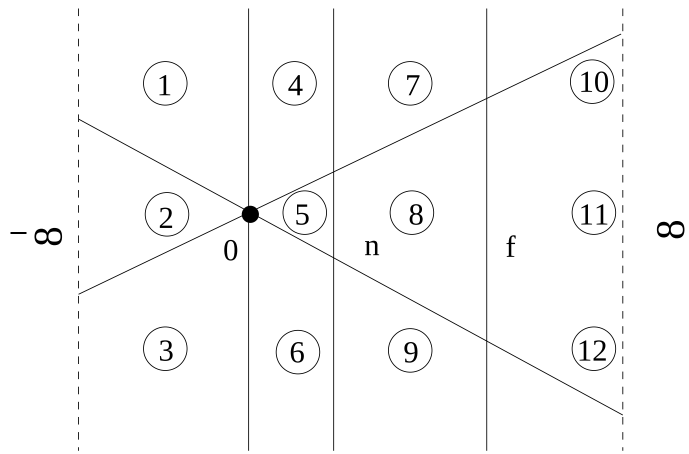
- Linear transformation

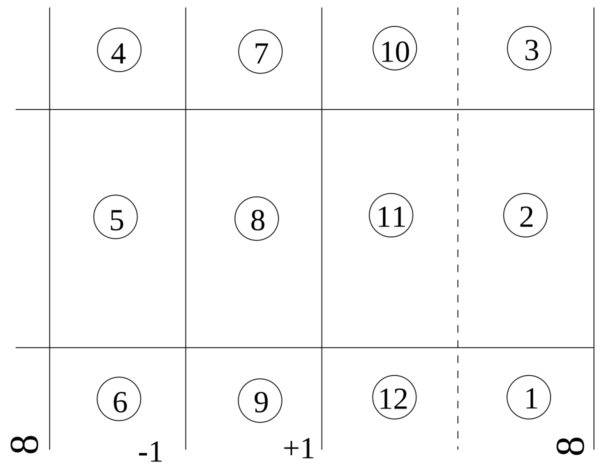
$$\begin{bmatrix} nr & 0 & 0 & 0 \\ 0 & ns & 0 & 0 \\ 0 & 0 & \frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \bar{w} \end{bmatrix}$$

- Normalization

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \bar{w} \end{bmatrix} = \begin{bmatrix} \bar{x}/\bar{w} \\ \bar{y}/\bar{w} \\ \bar{z}/\bar{w} \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

# Region Mapping



*Clipping not good after normalization:*

- Ambiguity after normalization

$$-1 \leq \frac{\bar{x}, \bar{y}, \bar{z}}{\bar{w}} \leq +1$$

  - Numerator can be positive or negative
  - Denominator can be positive or negative
- Normalization expended on points that are subsequently clipped.

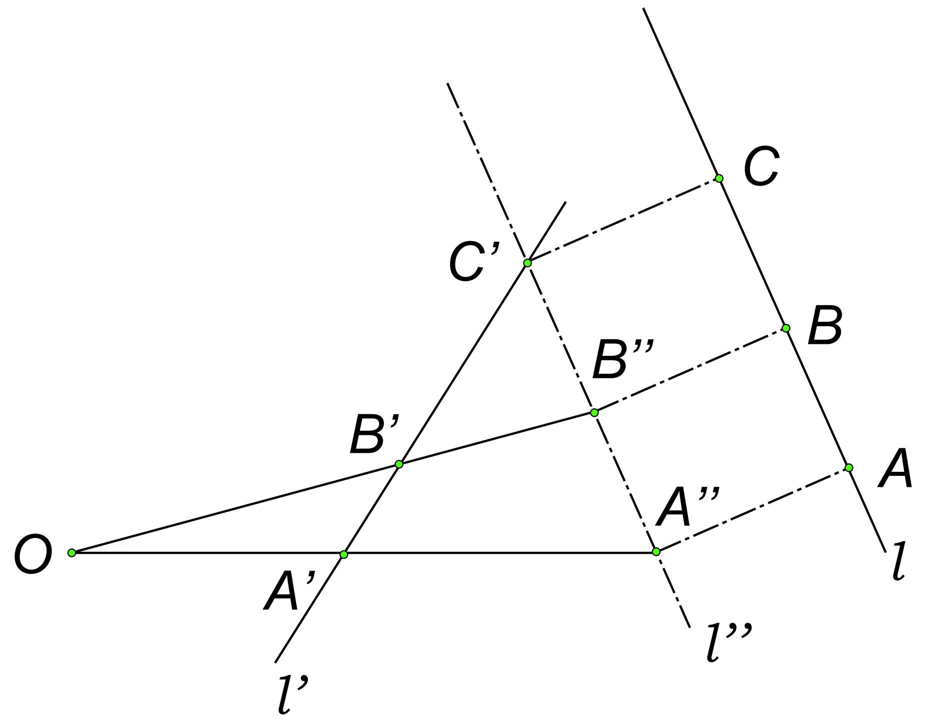
*Clipping in homogeneous coordinates:*

- Compare unnormalized coordinate against $\bar{w}$

$$-|\bar{w}| \leq \bar{x}, \bar{y}, \bar{z} \leq +|\bar{w}|$$

# Cross Ratio

*Definition:*




$$x = \frac{CA}{CB} \Big/ \frac{DA}{DB}$$

$$\frac{CA}{CB} \Big/ \frac{DA}{DB} = \frac{C'A'}{C'B'} \Big/ \frac{D'A'}{D'B'}$$

$$\text{area } OCA = \frac{1}{2} h \cdot CA = \frac{1}{2} OA \cdot OC \sin \angle COA$$

$$\text{area } OCB = \frac{1}{2} h \cdot CB = \frac{1}{2} OB \cdot OC \sin \angle COB$$

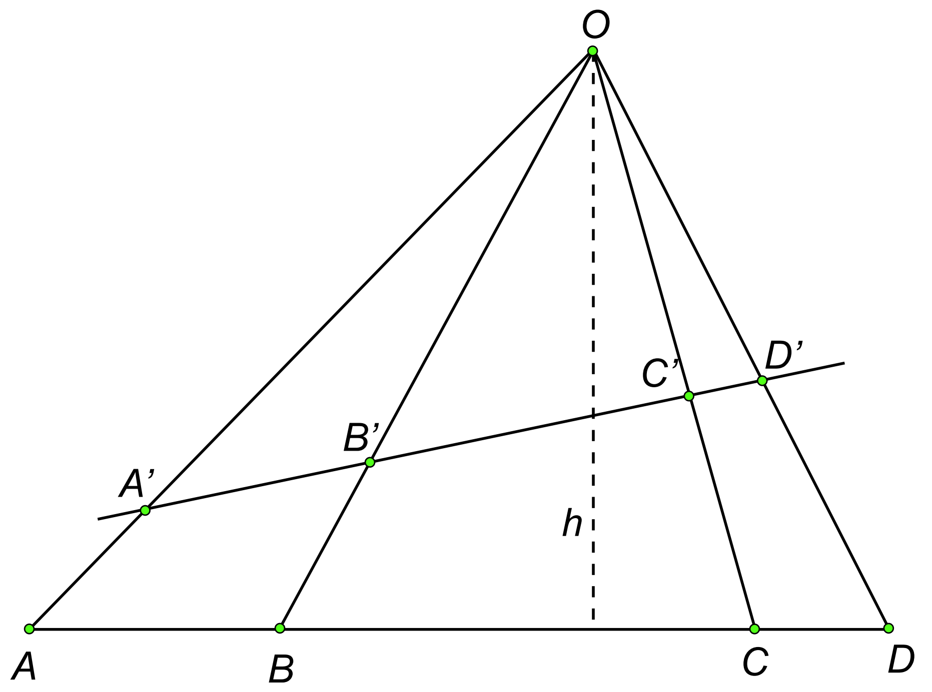
$$\text{area } ODA = \frac{1}{2} h \cdot DA = \frac{1}{2} OA \cdot OD \sin \angle DOA$$

$$\text{area } ODB = \frac{1}{2} h \cdot DB = \frac{1}{2} OB \cdot OD \sin \angle DOB$$

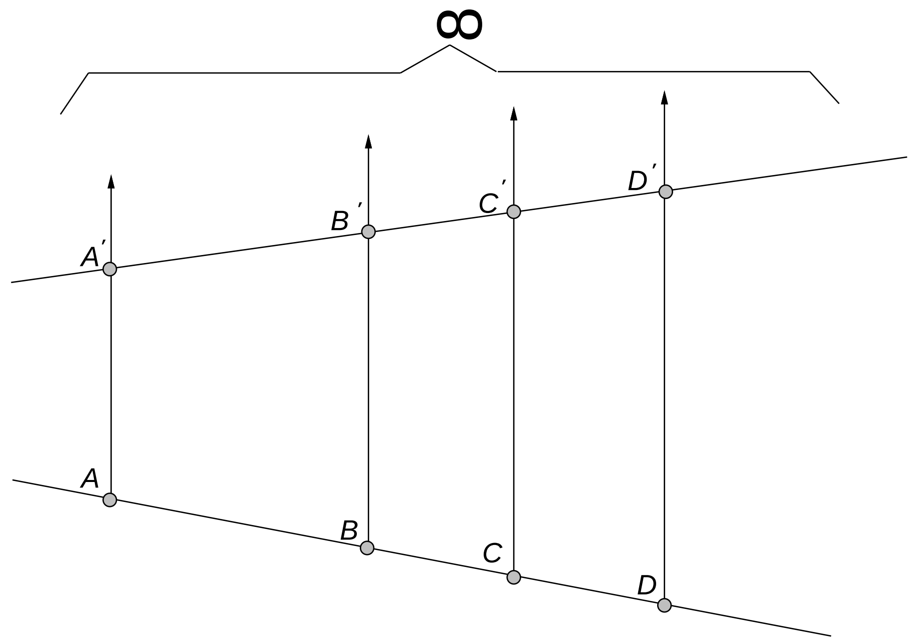
Hence

$$\frac{CA}{CB}\bigg/\frac{DA}{DB} = \frac{CA}{CB}\cdot\frac{DB}{DA} = \frac{OA\cdot OC\sin\angle COA}{OB\cdot OC\sin\angle COB}\cdot\frac{OB\cdot OD\sin\angle DOB}{OA\cdot OD\sin\angle DOA}$$

$$= \frac{\sin\angle COA}{\sin\angle COB}\cdot\frac{\sin\angle DOB}{\sin\angle DOA}$$

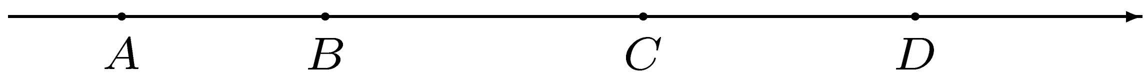



Invariance of cross-ratio under central projection

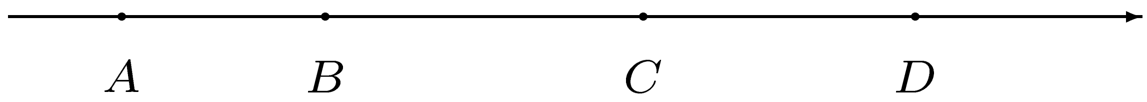



Invariance of cross-ratio under parallel projection

$(ABCD) > 0$



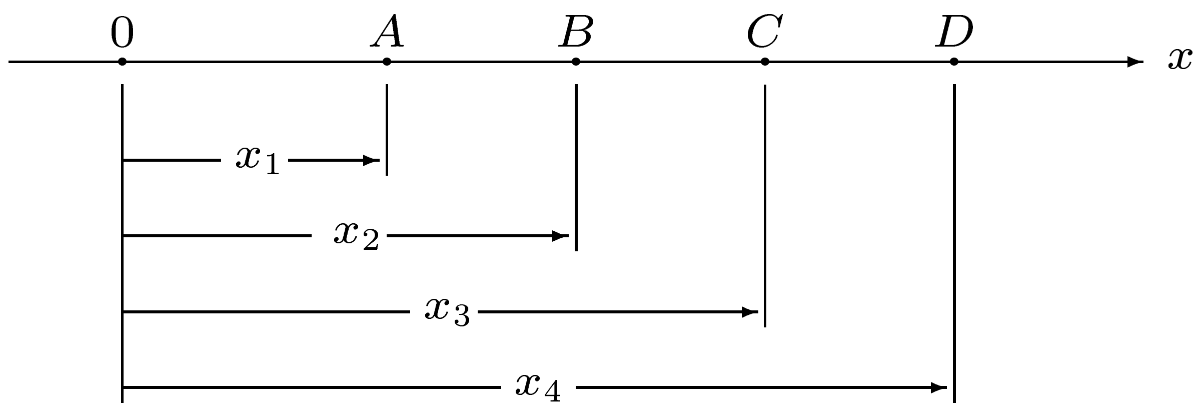

$(ABCD) < 0$




Sign of cross-ratio

$$(ABCD) = \frac{CA}{CB} \Big/ \frac{DA}{DB} = \frac{x_3 - x_1}{x_3 - x_2} \Big/ \frac{x_4 - x_1}{x_4 - x_2}$$

$$= \frac{x_3 - x_1}{x_3 - x_2} \cdot \frac{x_4 - x_2}{x_4 - x_1}$$




Cross-ratio in terms of coordinates.