Interpolating Unit Quaternions

To interpolate two orientations, we could use linear interpolation on all the entries of the quaternion (i.e. treat the quaternion as a four-dimensional vector). Unfortunately, intermediate quaternions would not have unit norm, and the angle velocity would not be constant even if we renormalized.

The solution: spherical linear interpolation, or the *slerp*. Consider the quaternions vectors, and find the angle between them:

$$\omega = \cos^{-1}(q_1 \cdot q_2).$$

Given a parameter $u \in [0, 1]$, the slerp interpolated value is defined as

$$q(u) = q_1 \frac{\sin((1-u)\omega)}{\sin(\omega)} + q_2 \frac{\sin(u\omega)}{\sin(\omega)}.$$

The slerp will have numerical difficulties when $\omega \approx 0$. In such a case, it is wise to replace the slerp with a lerp. There will also be a problem with $\omega \approx n\pi/2$; this should probably be flagged with an error and/or a request for more keyframes.

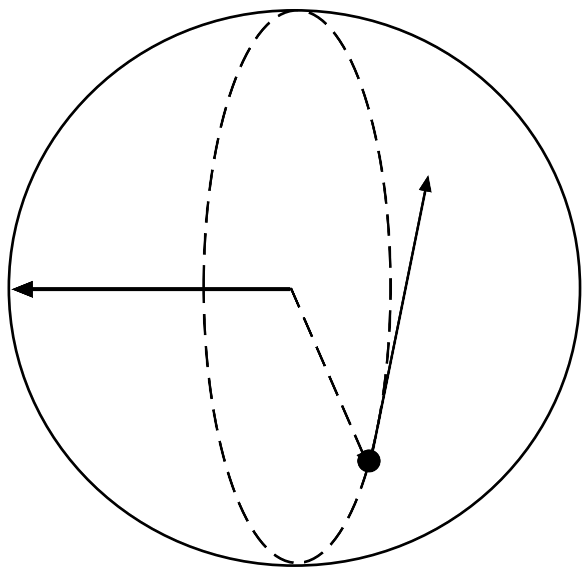
There are two possible great circles joining any two orientations. Normally, we want to choose the shortest one. This can be accomplished by testing the condition $(q_1 - q_2) \cdot (q_1 - q_2) > (q_1 + q_2) \cdot (q_1 + q_2)$. If true, q_2 should be replaced with $-q_2$. Recall that two quaternions that differ only in sign represent the same orientation.

3D Rotation User Interface

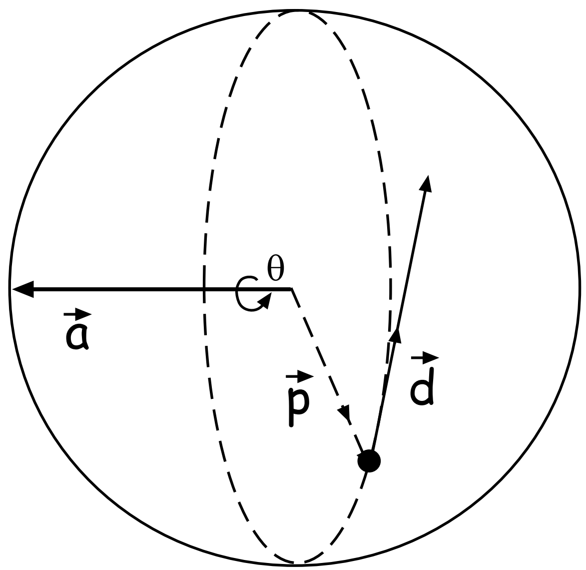
Goal: Want to specify angle-axis rotation "directly".

Problem: May only have mouse, which only has two degrees of freedom.

Solutions: Virtual Sphere, Arcball.

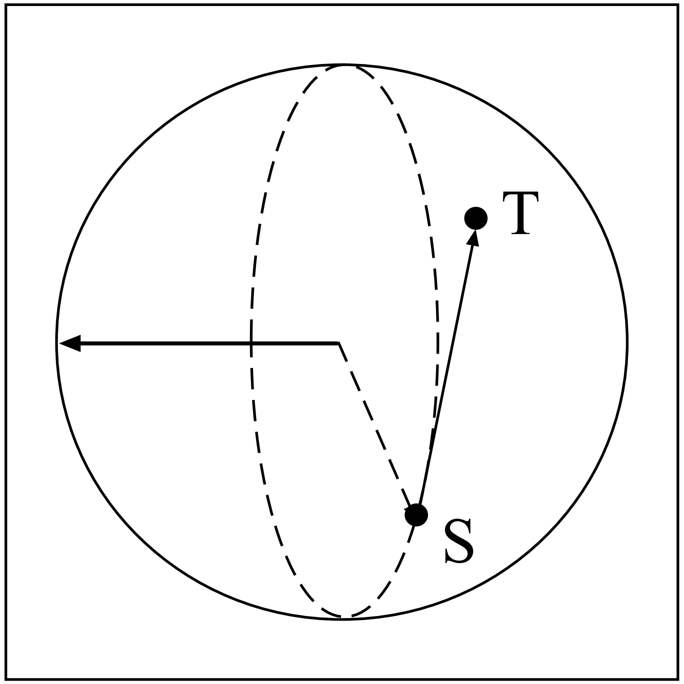


Virtual Sphere



Arcball

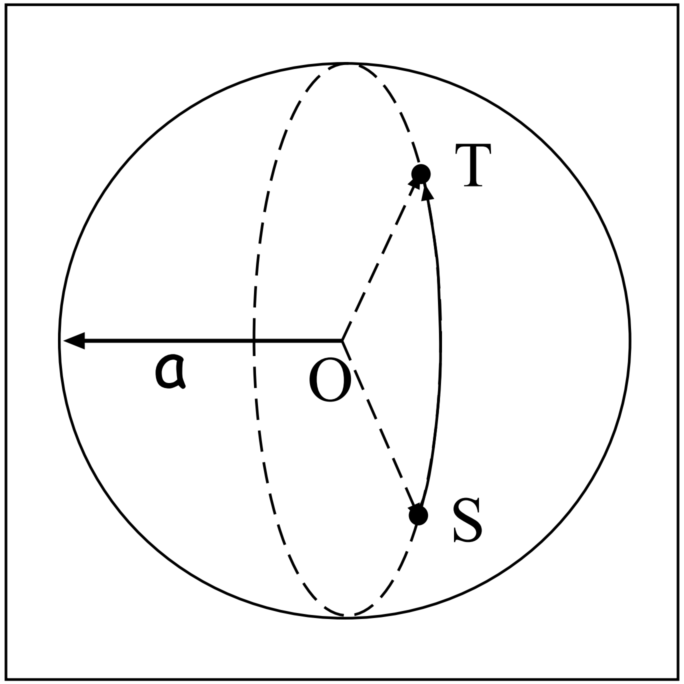
The Virtual Sphere

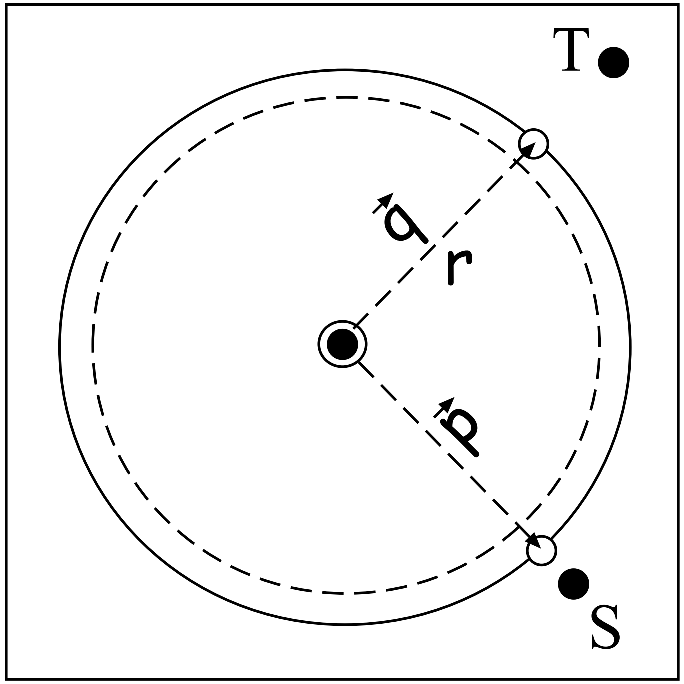


1. Define portion of screen to be projection of virtual sphere.
2. Get two sequential samples of mouse position, S and T.
3. Map 2D point S to 3D unit vector $\vec{p}$ on sphere.
4. Map 2D vector $\overrightarrow{ST}$ to 3D tangential velocity $\vec{d}$.

5. Normalize $\vec{d}$.
6. Axis: $\vec{a} = \vec{p} \times \vec{d}$.
7. Angle: $\theta = \alpha|\overrightarrow{ST}|$.
   (Choose α so a 180° rotation can be obtained.)
8. Save T to use as S for next time.

The Arcball





1. Choose region of sreen as projection of sphere: 2D point O center, radius ρ.
2. Get initial 2D point S on button-down.
3. Compute $\vec{s} = \overrightarrow{ST}$, $\vec{s} = (s_x, s_y)$.
4. Compute 2D radius: $r^2 = s_x^2 + s_y^2$.

5. Map 2D vector $\vec{s}$ to 3D unit vector $\vec{p}$:
   If $r^2 > \rho^2$, map to silhouette of unit sphere:

$$p_x \leftarrow s_x/r$$
$$p_y \leftarrow s_y/r$$
$$p_z \leftarrow 0.$$

   Else,

$$p_x \leftarrow s_x/\rho$$
$$p_y \leftarrow s_y/\rho$$
$$p_z \leftarrow \sqrt{1 - p_x^2 - p_y^2}.$$

6. For each new mouse position T, map to unit 3D vector $\vec{q}$ as above.
7. Axis: $\vec{a} = \vec{p} \times \vec{q}$.
8. Angle: $\theta = 2\cos^{-1}(\vec{p} \cdot \vec{q})$.

9. Notes:
   - Rotation given by *twice* the angle of the great arc between $\vec{p}$ and $\vec{q}$.
   - Doubling the angle matches orientation's mathematical structure better.
   - Points on opposite sides of the sphere silhouette allow a rotation by 360°.
     (so ambiguity of $\cos^{-1}(0)$ is finished . . . is always identity.)

Transforming Normals

- Normals are vectors perpendicular to all tangents at a point:

$$\vec{N} \cdot \vec{T} \equiv \mathbf{n}^T \mathbf{t} = 0.$$

- Note that the natural representation of $\vec{N}$ is as a *row vector*.
- Suppose we have a transformation M, a point $P \equiv \mathbf{p}$, and a tangent $\vec{T} \equiv \mathbf{t}$ at P.
- Let M_ℓ be the "linear part" of M, i.e. the upper 3×3 submatrix.

$$\begin{aligned} \mathbf{p}' &= M\mathbf{p} \\ \mathbf{t}' &= M\mathbf{t} \\ &= M_\ell \mathbf{t}. \\ \mathbf{n}^T\mathbf{t} &= \mathbf{n}^T M_\ell^{-1} M_\ell \mathbf{t} \\ &= (M_\ell^{-1T}\mathbf{n})^T (M_\ell \mathbf{t}) \\ &= (\mathbf{n}')^T \mathbf{t}' \end{aligned}$$

$$\equiv \vec{N}' \cdot \vec{T}'.$$

- Tranformation normals by inverse transpose of linear part of transformation: $\mathbf{n}' = M_\ell^{-1T}\mathbf{n}$.
- If M_T is orthogonal (usual case for rigid body transform), $M_T^{-1T} = M_T$.

Only worry if you have a shear transformation.