Conic Sections and Splines

- Conics
  - Implicit form
  - Parametric form
- Rational Bézier Forms
  - General quadratics
  - Conversion forms for conics
- Conics as Parametric rational Bézier form

Conic Curves

Conic Sections (Implicit form)

- Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a, b > 0$$

- Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad a, b > 0$$

- Parabola

$$y^2 = 4ax \quad a > 0$$

Conic Sections (Parametric form)

- Ellipse

$$\begin{aligned} x(t) &= a\frac{1-t^2}{1+t^2} \\ y(t) &= b\frac{2t}{1+t^2} \quad (-\infty < t < +\infty) \end{aligned}$$

- Hyperbola

$$\begin{aligned} x(t) &= a\frac{1+t^2}{1-t^2} \\ y(t) &= b\frac{2t}{1-t^2} \quad (-\infty < t < +\infty) \end{aligned}$$

- Parabola

$$
\begin{aligned}
x(t) &= at^2 \\
y(t) &= 2at \quad (-\infty < t < +\infty)
\end{aligned}
$$

Rational Quadratic Bezier Forms

Quadratic Rational Bézier Form:

- Homogeneous form

$$\begin{bmatrix} x(t) \\ y(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ w_0 \end{bmatrix} B_0^2(t) + \begin{bmatrix} x_1 \\ y_1 \\ w_1 \end{bmatrix} B_1^2(t) + \begin{bmatrix} x_2 \\ y_2 \\ w_2 \end{bmatrix} B_2^2(t)$$

$$= \begin{bmatrix} x_0 B_0^2(t) + x_1 B_1^2(t) + x_2 B_2^2(t) \\ y_0 B_0^2(t) + y_1 B_1^2(t) + y_2 B_2^2(t) \\ w_0 B_0^2(t) + w_1 B_1^2(t) + w_2 B_2^2(t) \end{bmatrix}$$

- Rational (projected) form

$$\begin{bmatrix} \bar{x}(t) \\ \bar{y}(t) \end{bmatrix} = \begin{bmatrix} \frac{x_0 B_0^2(t) + x_1 B_1^2(t) + x_2 B_2^2(t)}{w_0 B_0^2(t) + w_1 B_1^2(t) + w_2 B_2^2(t)} \\ \frac{y_0 B_0^2(t) + y_1 B_1^2(t) + y_2 B_2^2(t)}{w_0 B_0^2(t) + w_1 B_1^2(t) + w_2 B_2^2(t)} \end{bmatrix}$$

$$= \frac{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} B_0^2(t) + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} B_1^2(t) + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} B_2^2(t)}{w_0 B_0^2(t) + w_1 B_1^2(t) + w_2 B_2^2(t)}$$

Conversions:

- Conic parameterization elements in Bézier form

$$\begin{aligned} 2t &= B_1^2(t) + 2B_2^2(t) \\ 1 - t^2 &= B_0^2(t) + B_1^2(t) \\ 1 + t^2 &= B_0^2(t) + B_1^2(t) + 2B_2^2(t) \\ t^2 &= B_2^2(t) \end{aligned}$$

Conics as Rational Bézier Curves

Conics as Rational Bézier (Ellipse)

- Rational Bézier

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{\begin{bmatrix} a(1-t^2) \\ b(2t) \end{bmatrix}}{1+t^2}$$

$$= \frac{\begin{bmatrix} aB_0^2(t) + aB_1^2(t) + 0B_2^2(t) \\ 0B_0^2(t) + bB_1^2(t) + b2B_2^2(t) \end{bmatrix}}{B_0^2(t) + B_1^2(t) + 2B_2^2(t)}$$

$$= \frac{\begin{bmatrix} a \\ 0 \end{bmatrix} B_0^2(t) + \begin{bmatrix} a \\ b \end{bmatrix} B_1^2(t) + \begin{bmatrix} 0 \\ 2b \end{bmatrix} B_2^2(t)}{B_0^2(t) + B_1^2(t) + 2B_2^2(t)}$$

which implies

$$\begin{array}{lll} w_0 = 1 & x_0 = a & y_0 = 0 \\ w_1 = 1 & x_1 = a & y_1 = b \\ w_2 = 2 & x_2 = 0 & y_2 = 2b \end{array}$$

Conics as Rational Bézier (Hyperbola)

- Rational Bézier

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{\begin{bmatrix} a(1+t^2) \\ b(2t) \end{bmatrix}}{1-t^2}$$

$$= \frac{\begin{bmatrix} aB_0^2(t) + aB_1^2(t) + a2B_2^2(t) \\ 0B_0^2(t) + bB_1^2(t) + b2B_2^2(t) \end{bmatrix}}{B_0^2(t) + B_1^2(t)}$$

$$= \frac{\begin{bmatrix} a \\ 0 \end{bmatrix} B_0^2(t) + \begin{bmatrix} a \\ b \end{bmatrix} B_1^2(t) + \begin{bmatrix} 2a \\ 2b \end{bmatrix} B_2^2(t)}{B_0^2(t) + B_1^2(t) + 0B_2^2(t)}$$

which implies

$$\begin{array}{lll} w_0 = 1 & x_0 = a & y_0 = 0 \\ w_1 = 1 & x_1 = a & y_1 = b \\ w_2 = 0 & x_2 = 2a & y_2 = 2b \end{array}$$

Conics as Rational Bézier (Parabola)

- Rational Bézier

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{\begin{bmatrix} a(t^2) \\ a(2t) \end{bmatrix}}{1}$$

$$= \frac{\begin{bmatrix} 0B_0^2(t) + 0B_1^2(t) + aB_2^2(t) \\ 0B_0^2(t) + aB_1^2(t) + a2B_2^2(t) \end{bmatrix}}{B_0^2(t) + B_1^2(t) + B_2^2(t)}$$

$$= \frac{\begin{bmatrix} 0 \\ 0 \end{bmatrix} B_0^2(t) + \begin{bmatrix} 0 \\ a \end{bmatrix} B_1^2(t) + \begin{bmatrix} a \\ 2a \end{bmatrix} B_2^2(t)}{B_0^2(t) + B_1^2(t) + B_2^2(t)}$$

which implies

$$\begin{array}{lll} w_0 = 1 & x_0 = 0 & y_0 = 0 \\ w_1 = 1 & x_1 = 0 & y_1 = a \\ w_2 = 1 & x_2 = a & y_2 = 2a \end{array}$$

Not Unique

- x, y, w are not unique
  - Numerator and denominator can be multiplied by a common (positive) factor
- The following example is a common alternative form:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} w_0 B_0^2(t) + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} w_1 B_1^2(t) + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} w_2 B_2^2(t)}{w_0 B_0^2(t) + w_1 B_1^2(t) + w_2 B_2^2(t)}$$

which derives from rewriting

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \longrightarrow w \begin{bmatrix} \bar{x} \\ \bar{y} \\ 1 \end{bmatrix}$$

Bernstein Basis Functions

Bernstein Polynomial Properties:

Partition of Unity: $\sum_{i=0}^{n} B_i^n(t) = 1$

Proof:

$$\begin{aligned} 1 &= (t + (1-t))^n \\ &= \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i \\ &= \sum_{i=0}^{n} B_i^n(t) \end{aligned}$$

Nonnegativity: $B_i^n(t) \geq 0$, for $t \in [0, 1]$

Proof:

$$\begin{aligned} \binom{n}{i} &> 0 \\ t &\geq 0 \quad \text{for} \quad 0 \leq t \leq 1 \\ (1-t) &\geq 0 \quad \text{for} \quad 0 \leq t \leq 1 \end{aligned}$$

Recurrence: $B_0^0(t) = 1$ and $B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$

Proof:

$$\begin{aligned}
B_i^n(t) &= \binom{n}{i} t^i (1-t)^{n-i} \\
&= \binom{n-1}{i} t^i (1-t)^{n-i} + \binom{n-1}{i-1} t^i (1-t)^{n-i} \\
&= (1-t)\binom{n-1}{i} t^i (1-t)^{(n-1)-i} + t\binom{n-1}{i-1} t^{i-1} (1-t)^{(n-1)-(i-1)} \\
&= (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)
\end{aligned}$$

Derivatives: $\frac{d}{dt}B_i^n(t) = n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)$

Proof:

$$\begin{aligned}
\frac{d}{dt}B_i^n(t) &= \frac{d}{dt}\binom{n}{i}t^i(1-t)^{n-i} \\
&= \frac{d}{dt}\frac{n!}{i!(n-i)!}t^i(1-t)^{n-i} \\
&= \frac{in!}{i!(n-i)!}t^{i-1}(1-t)^{n-i} - \frac{(n-i)n!}{i!(n-i)!}t^i(1-t)^{n-i-1} \\
&= \frac{n(n-1)!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i} - \frac{n(n-1)!}{i!(n-i-1)!}t^i(1-t)^{n-i-1} \\
&= n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)
\end{aligned}$$

Bézier Splines

Bézier Curve Segments and their Properties

Definition:

- A degree n (order $n+1$) *Bézier curve segment* is

$$P(t) = \sum_{i=0}^{n} \vec{P}_i B_i^n(t)$$

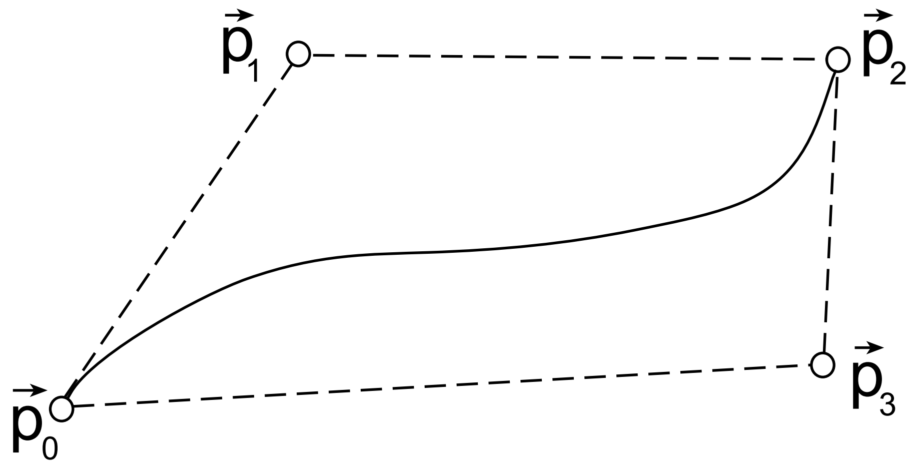
where the $\vec{P}_i$ are k-dimensional *control points*.

Convex Hull:

$\sum_{i=0}^{n} B_i^n(t) = 1$, $B_i^n(t) \geq 0$ for $t \in [0,1]$

$\Longrightarrow P(t)$ is a convex combination of the $\vec{P}_i$ for $t \in [0,1]$

$\Longrightarrow P(t)$ lies within convex hull of $\vec{P}_i$ for $t \in [0, 1]$



Affine Invariance:

- A Bézier curve is an affine combination of its control points
- Any affine transformation of a curve is the curve of the transformed control points

$$T\left(\sum_{i=0}^{n} P_i B_i^n(t)\right) = \sum_{i=0}^{n} T(P_i) B_i^n(t)$$

- *This property does not hold for projective transformations!*

Interpolation:

$B_0^n(0) = 1, B_n^n(1) = 1, \sum_{i=0}^{n} B_i^n(t) = 1, B_i^n(t) \geq 0$ for $t \in [0, 1]$

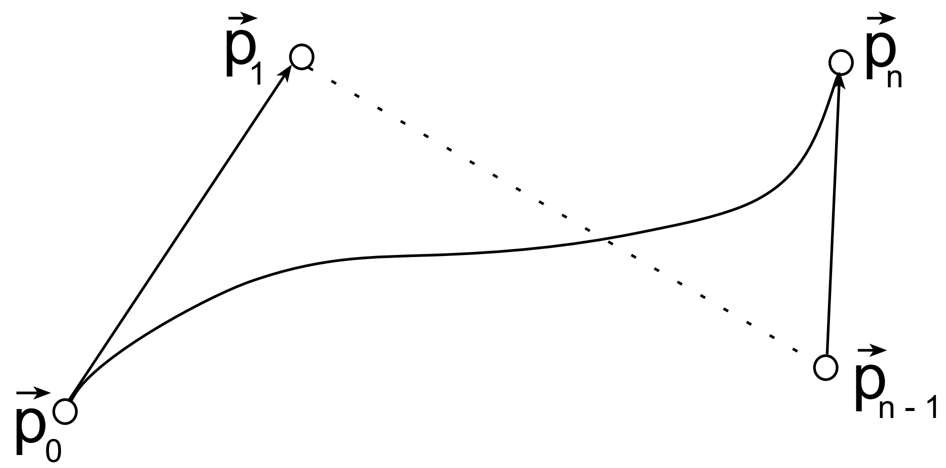
$\Longrightarrow B_i^n(0) = 0$ if $i \neq 0, B_i^n(1) = 0$ if $i \neq n$

$\Longrightarrow P(0) = P_0, P(1) = P_n$

Derivatives:

$$\frac{d}{dt}B_i^n(t) = n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)$$

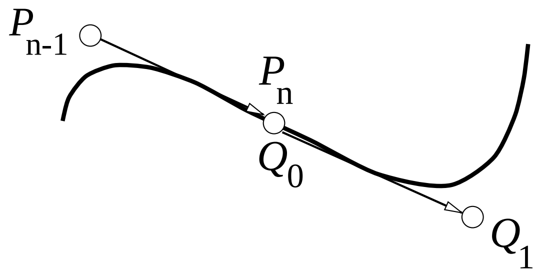
$$\Longrightarrow P'(0) = n(\vec{P}_1 - \vec{P}_0), P'(1) = n(\vec{P}_n - \vec{P}_{n-1})$$



Smoothly Joined Segments (G^1):

- Let P_{n-1}, P_n be the last two control points of one segment
- Let Q_0, Q_1 be the first two control points of the next segment

$$\begin{aligned} P_n &= Q_0 \\ (P_n - P_{n-1}) &= \beta(Q_1 - Q_0) \text{ for some } \beta > 0 \end{aligned}$$



Matrix View:

- Expand each Bernstein polynomial in powers of t
- Represent each expansion as the column of a matrix
- Quadratic example:

$$(1-t)^2 P_0 + 2(1-t)tP_1 + t^2 P_2 = (1-2t+t^2)P_0 + (2t-2t^2)P_1 + t^2 P_2$$

$$= [1\ t\ t^2] \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$$

- In matrix format:

$$P(t) = T(t)^T M_{BT} P$$

  - $T(t)^T = [1\ t\ t^2]$ is the *monomial basis*
  - $P_T = M_{BT}P$ is a matrix containing the coefficients of the polynomials for each dimension of $P(t)$
  - M_{BT} is a *change of basis matrix* that converts a specification P of $P(t)$ relative to the Bernstein basis to one relative to the monomial basis

Tensor Product Patches

Tensor Product Patches:

- The *control polygon* is the polygonal mesh with vertices $P_{i,j}$
- The *patch basis functions* are products of curve basis functions

$$P(s,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} P_{i,j} B_{i,j}^{n}(s,t)$$

where

$$B_{i,j}^{n}(s,t) = B_{i}^{n}(s) B_{j}^{n}(t)$$

Scan in image.

Properties:

- Patch basis functions *sume to one*

$$\sum_{i=0}^{n}\sum_{j=0}^{n} B_i^n(s)B_j^n(t) = 1$$

- Patch basis functions are *nonnegative* on [0, 1] × [0, 1]

$$B_i^n(s)B_j^n(t) \geq 0 \text{ for } 0 \leq s, t \leq 1$$

$\Longrightarrow$ Surface patch is in the *convex hull* of the control points
$\Longrightarrow$ Surface patch is *affinely invariant*
(Transform the patch by transforming the control points)

Subdivision, Recursion, Evaluation:

- As for curves in each variable separately and independently
- Normals must be computed from partial derivatives

Partial Derivatives:

- Ordinary derivative in each variable separately':

$$\frac{\partial}{\partial s}P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} P_{i,j}\left[\frac{d}{ds}B_i^n(s)\right]B_j^n(t)$$

$$\frac{\partial}{\partial t}P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} P_{i,j}B_i^n(s)\left[\frac{d}{dt}B_j^n(t)\right]$$

- Each of the above is a *tangent vector* in a parametric direction
- Surface is *regular* at each (s,t) where these two vectors are linearly independent
- The (unnormalized) *surface normal* is given at any regular point by

$$\pm\left[\frac{\partial}{\partial s}P(s,t)\times\frac{\partial}{\partial t}P(s,t)\right]$$

(the sign dictates what is the *outward pointing normal*)

- In particular, the *cross-boundary tangent* is given by (e.g., for the $s = 0$ boundary):

$$n \sum_{i=0}^{n} \sum_{j=0}^{n} (P_{1,j} - P_{0,j}) B_j^n(t)$$

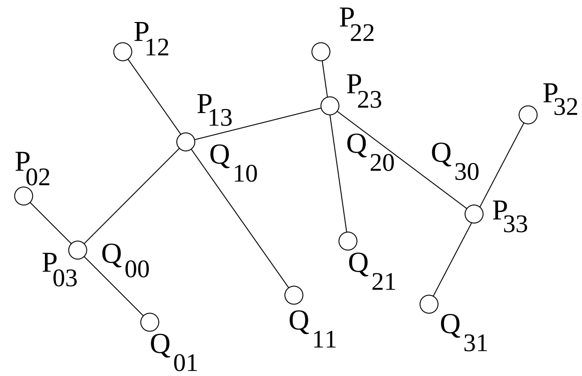
(and similarly for the other boundaries)

Smoothly Joined Patches:

- Can be achieved by ensuring that

$$(P_{i,n} - P_{i,n-1}) = \beta(Q_{i,1} - Qi, 0) \text{ for } \beta > 0$$

(and correspondingly for other boundaries)



Rendering:

- Divide up into polygons:
  1. By stepping

$$
\begin{aligned}
s &= 0, \delta, 2\delta, \ldots, 1 \\
t &= 1, \gamma, 2\gamma, \ldots, 1
\end{aligned}
$$

     and joining up sides and diagonals to produce a triangular mesh
  2. By subdividing and rendering the control polygon