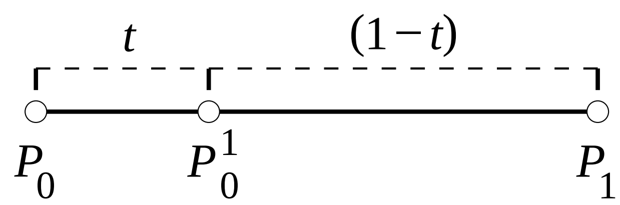
Constructing Curve Segments

Linear blend:

- Line segment from an affine combination of points

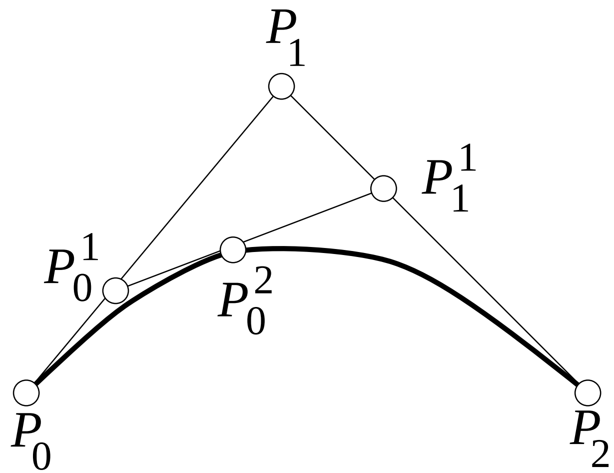
$$P_0^1(t) = (1 - t)P_0 + tP_1$$



Quadratic blend:

- Quadratic segment from an affine combination of line segments

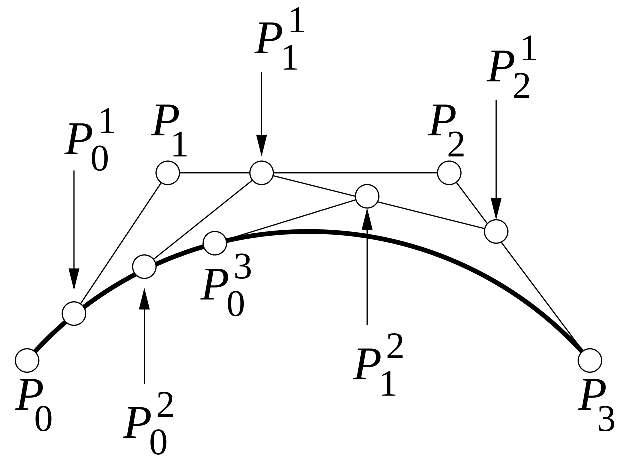
$$
\begin{aligned}
P_0^1(t) &= (1-t)P_0 + tP_1 \\
P_1^1(t) &= (1-t)P_1 + tP_2 \\
P_0^2(t) &= (1-t)P_0^1(t) + tP_1^1(t)
\end{aligned}
$$



Cubic blend:

- Cubic segment from an affine combination of quadratic segments

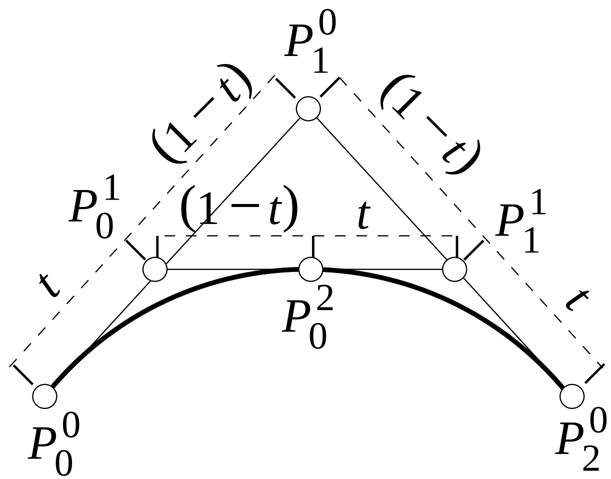
$$\begin{aligned}
P_0^1(t) &= (1-t)P_0 + tP_1 \\
P_1^1(t) &= (1-t)P_1 + tP_2 \\
P_0^2(t) &= (1-t)P_0^1(t) + tP_1^1(t) \\
P_1^1(t) &= (1-t)P_1 + tP_2 \\
P_2^1(t) &= (1-t)P_2 + tP_3 \\
P_1^2(t) &= (1-t)P_1^1(t) + tP_2^1(t) \\
P_0^3(t) &= (1-t)P_0^2(t) + tP_1^2(t)
\end{aligned}$$



- The pattern should be evident for higher degrees

Geometric view (de Casteljau Algorithm):

- Join the points P_i by line segments
- Join the $t : (1 - t)$ points of those line segments by line segments
- Repeat as necessary
- The $t : (1 - t)$ point on the final line segment is a point on the curve
- The final line segment is tangent to the curve at t



Expanding Terms (Basis Polynomials):

- The original points appear as coefficients of *Bernstein polynomials*

$$
\begin{aligned}
P_0^0(t) &= P_0 1 \\
P_0^1(t) &= (1-t)P_0 + tP_1 \\
P_0^2(t) &= (1-t)^2 P_0 + 2(1-t)tP_1 + t^2 P_2 \\
P_0^3(t) &= (1-t)^3 P_0 + 3(1-t)^2 tP_1 + 3(1-t)t^2 P_2 + t^3 P_3 \\
P_0^n(t) &= \sum_{i=0}^{n} P_i B_i^n(t)
\end{aligned}
$$

where $B_i^n(t) = \frac{n!}{(n-i)!i!}(1-t)^{n-i}t^i = \binom{n}{i}(1-t)^{n-i}t^i$

- The Bernstein polynomials of degree n form a basis for the space of all degree-n polynomials

Recursive evaluation schemes:

- To obtain curve points:
  - Start with given points and form successive, pairwise, affine combinations

$$\begin{aligned} P_i^0 &= P_i \\ P_i^j &= (1-t)P_i^{j-1} + tP_{i+1}^{j-1} \end{aligned}$$

  - The generated points P_i^j are the *deCasteljau points*
- To obtain basis polynomials:
  - Start with 1 and form successive, pairwise, affine combinations

$$\begin{aligned} B_0^0 &= 1 \\ B_i^j &= (1-t)B_i^{j-1} + tB_{i+1}^{j-1} \end{aligned}$$

where $B_r^s = 0$ when $r < 0$ or $r > s$

Recursive triangle diagrams (upward):

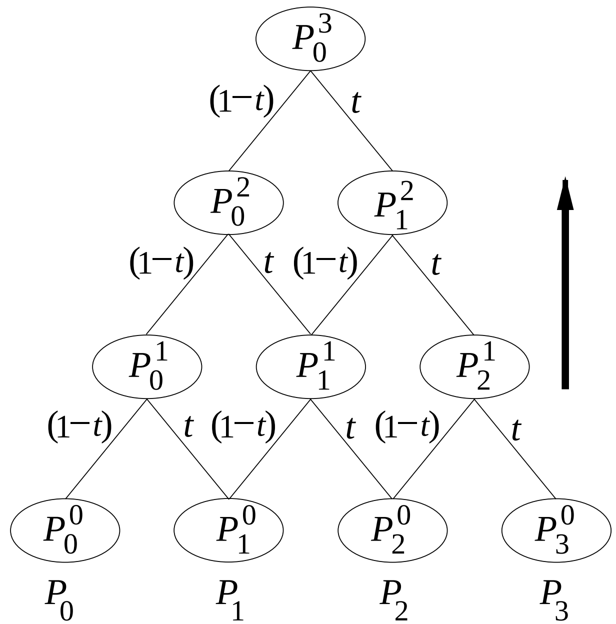
Computing deCasteljau points

- Each node gets the affine combination of the two nodes entering from below
  - Leaf nodes have the value of their respective points

$$P_1^2 = (1-t)P_1^1 + tP_2^1$$

- Each node gets the sum of the path products entering from below

$$\begin{aligned} P_1^2 &= P_0^1(1-t)(1-t) + P_0^2 t(1-t) + P_0^2(1-t)t + P_3^0 tt \\ \Rightarrow P_1^2 &= (1-t)^2 P_0^1 + 2(1-t)tP_0^2 + t^2 P_3^0 \end{aligned}$$



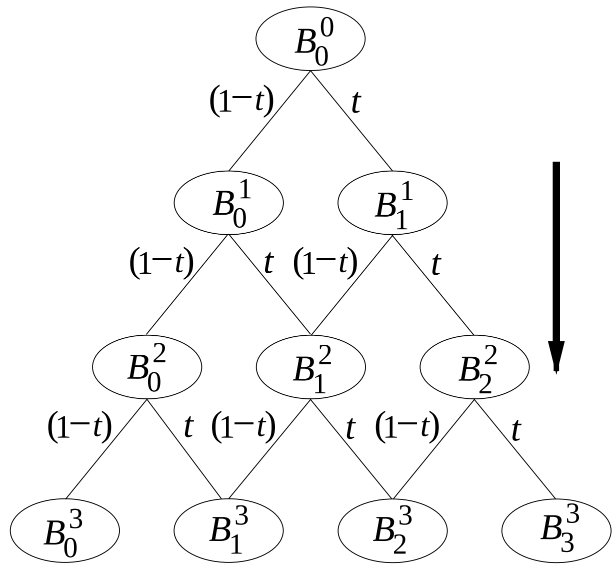
P_0^3
(1−t) t
P_0^2 P_1^2
(1−t) t (1−t) t
P_0^1 P_1^1 P_2^1
(1−t) t (1−t) t (1−t) t
P_0^0 P_1^0 P_2^0 P_3^0
P_0 P_1 P_2 P_3

Recursive triangle diagrams (downward):

Computing Bernstein (basis) polynomials

- Each node gets the affine combination of the two nodes entering from above
  - Root node has value 1
  - For other nodes, missing entries above count as zero
- Each node gets the sum of the path products entering from above

$$
\begin{aligned}
B_1^3 &= t(1-t)(1-t) + (1-t)t(1-t) + (1-t)t(1-t)t \\
\Rightarrow P_1^3 &= 3(1-t)^2 t
\end{aligned}
$$



B_0^0
(1−t)
t
B_0^1
B_1^1
(1−t)
t
(1−t)
t
B_0^2
B_1^2
B_2^2
(1−t)
t
(1−t)
t
(1−t)
t
B_0^3
B_1^3
B_2^3
B_3^3

Recurrence, Subdivision:

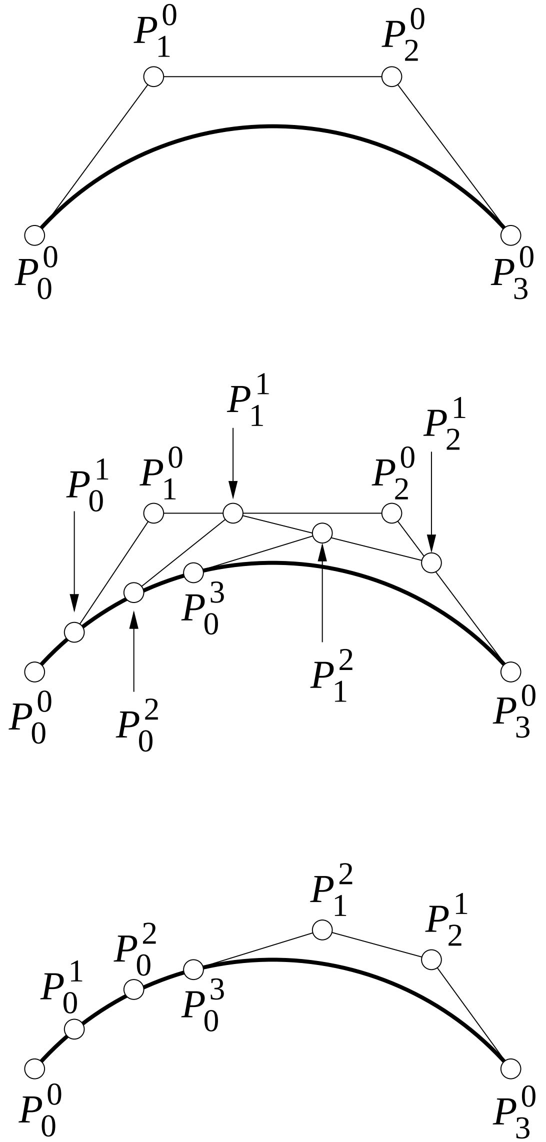
$$B_i^n(t) = (1-t)B_i^{n-1} + tB_{i-1}^{n-1}(t)$$

$\Longrightarrow$ deCasteljau's algorithm:

$$\begin{aligned} P(t) &= P_o^n(t) \\ P_i^k(t) &= (1-t)P_i^{k-1}(t) + t)P_{i+1}^{k-1} \\ P_i^0 &= P_i \end{aligned}$$

Use to evaluate point at t, or subdivide into two new curves:

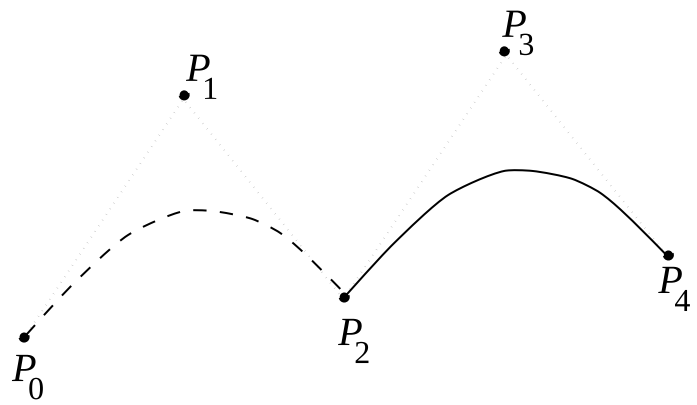
- $P_0^0, P_0^1, \ldots P_0^n$ are the control points for the left half
- $P_n^0, P_{n-1}^1, \ldots P_0^n$ are the control points for the right half


P_1^0
P_2^0
P_0^0
P_3^0
P_1^1
P_2^1
P_0^1
P_1^0
P_2^0
P_0^3
P_1^2
P_0^0
P_0^2
P_3^0
P_1^2
P_2^1
P_0^2
P_0^1
P_0^3
P_0^0
P_3^0

Discontinuities in Splines

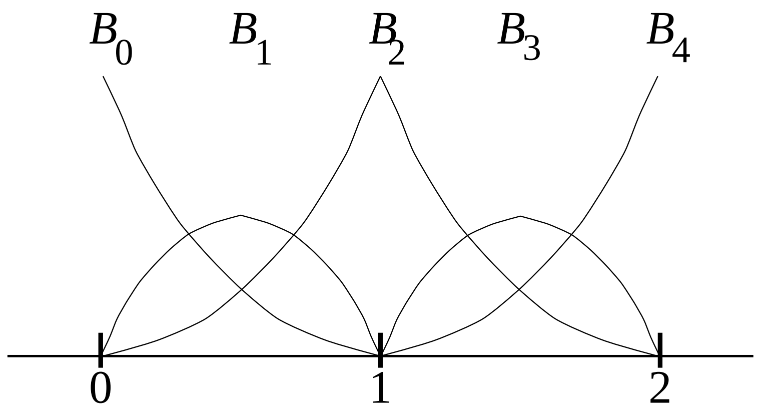
Bézier Discontinuities:

- Two Bézier segments can be completely disjoint
- Two segments join if they share last/first control point



Common Parameterization and Blending Functions

- Joined curves can be given common parameterization
  - Parameterize first segment with $0 \leq t < 1$
  - Parameterize nest segment with $1 \leq t \leq 2$, etc.
- Look at blending/basis polynomials under this parameterization
  - Combine those for common P_j into a single piecewise polynomial



Combined Curve Segments

- Curve is $P(t) = P_0B_0(t) + P_1B_1(t) + P_2B_2(t) + P_3B_3(t) + P_4B_4(t)$, where

$$B_0(t) = \begin{cases} (1-t)^2 & 0 \le t < 1 \\ 0 & 1 \le t \le 2 \end{cases}$$

$$B_1(t) = \begin{cases} 2(1-t)t & 0 \le t < 1 \\ 0 & 1 \le t \le 2 \end{cases}$$

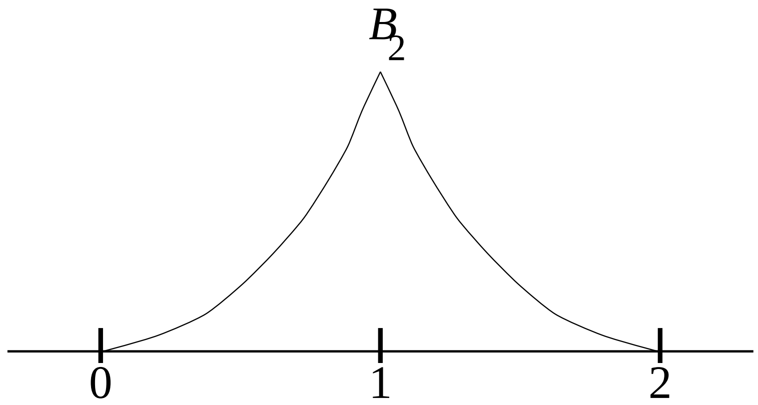
$$B_2(t) = \begin{cases} t^2 & 0 \le t < 1 \\ (2-t)^2 & 1 \le t \le 2 \end{cases}$$

$$B_3(t) = \begin{cases} 0 & 0 \le t < 1 \\ 2(2-t)(t-1) & 1 \le t \le 2 \end{cases}$$

$$B_4(t) = \begin{cases} 0 & 0 \le t < 1 \\ (t-1)^2 & 1 \le t \le 2 \end{cases}$$

Curve Discontinuities from Basis Discontinuities

- P_2 is scaled by $B_2(t)$, which has a discontinuous derivative
- The corner in the curve results from this discontinuity



Spline Continuity

Smoother Blending Functions:

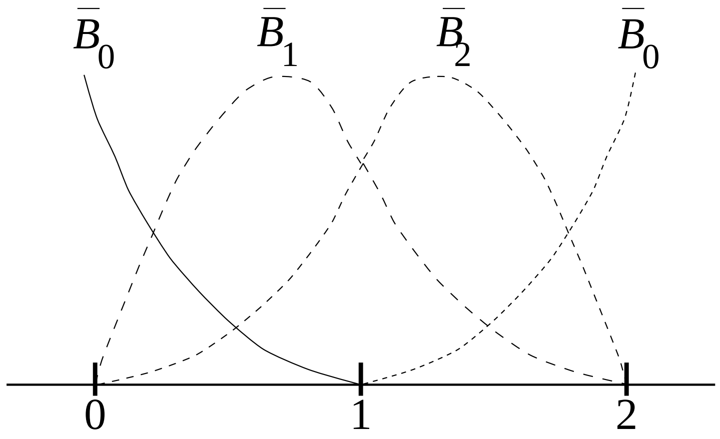
- Can $B_0(t), \ldots, B_4(t)$ be replaced by smoother functions?
  - Piecewise polynomials on $0 \leq t \leq 2$
  - Continuous derivatives
- Yes, but we lose one degree of freedom
  - Curve has no corner if segments share a common tangent
  - Tangent is given by the chords $\overline{P_1 P_2}$, $\overline{P_2 P_3}$
  - An equation constrains P_1, P_2, P_3
    $P_3 - P_2 = P_2 - P_1 \Longrightarrow P_2 = \frac{P_1 + P_3}{2}$
- This equation leads to combinations:

$$P_0 B_0(t) + P_1 \left(B_1(t) + \frac{1}{2} B_2(t) \right) + P_3 \left(\frac{1}{2} B_2(t) + B_3(t) \right) + P_4 B_4(t)$$

Spline Basis:

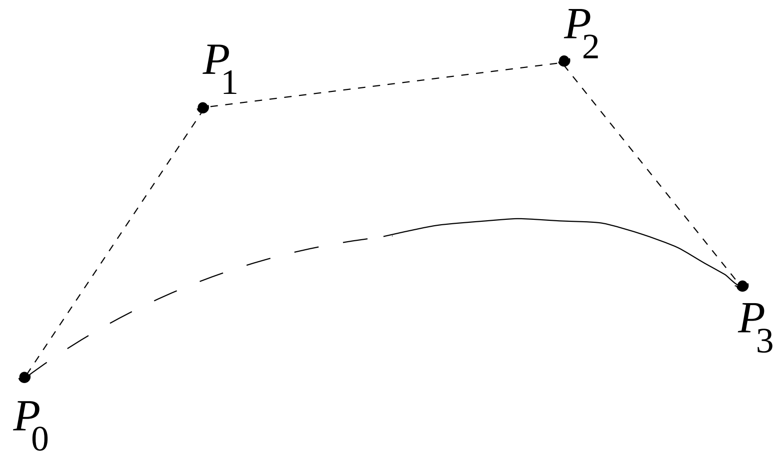
- Combined functions form a smoother *spline basis*

$$\begin{aligned}\overline{B}_0(t) &= B_0(t)\\ \overline{B}_1(t) &= \left(B_1(t) + \frac{1}{2}B_2(t)\right)\\ \overline{B}_2(t) &= \left(\frac{1}{2}B_2(t) + B_3(t)\right)\\ \overline{B}3_(t) &= B_4(t)\end{aligned}$$


$\bar{B}_0$
$\bar{B}_1$
$\bar{B}_2$
$\bar{B}_0$
0
1
2

Smoother Curves:

- Control points used with this basis produce smoother curves.

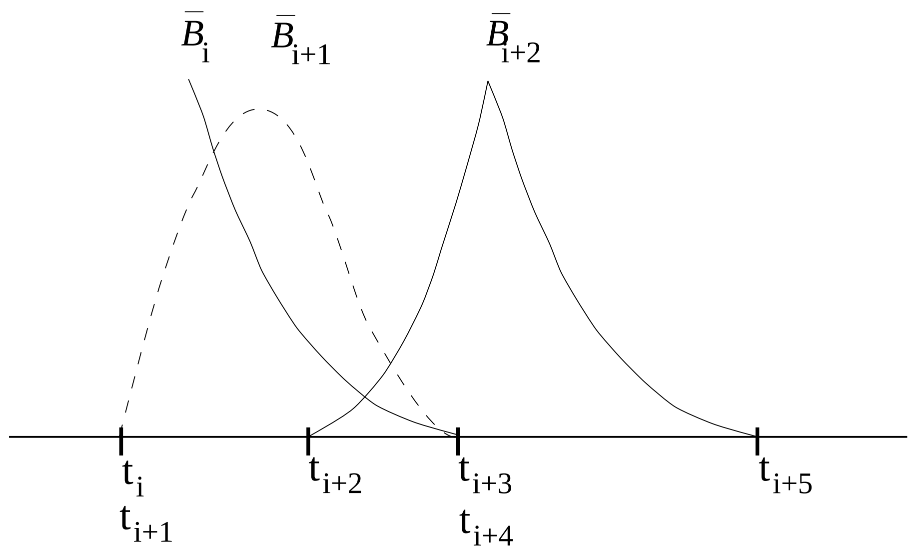


B-Splines

General B-Splines:

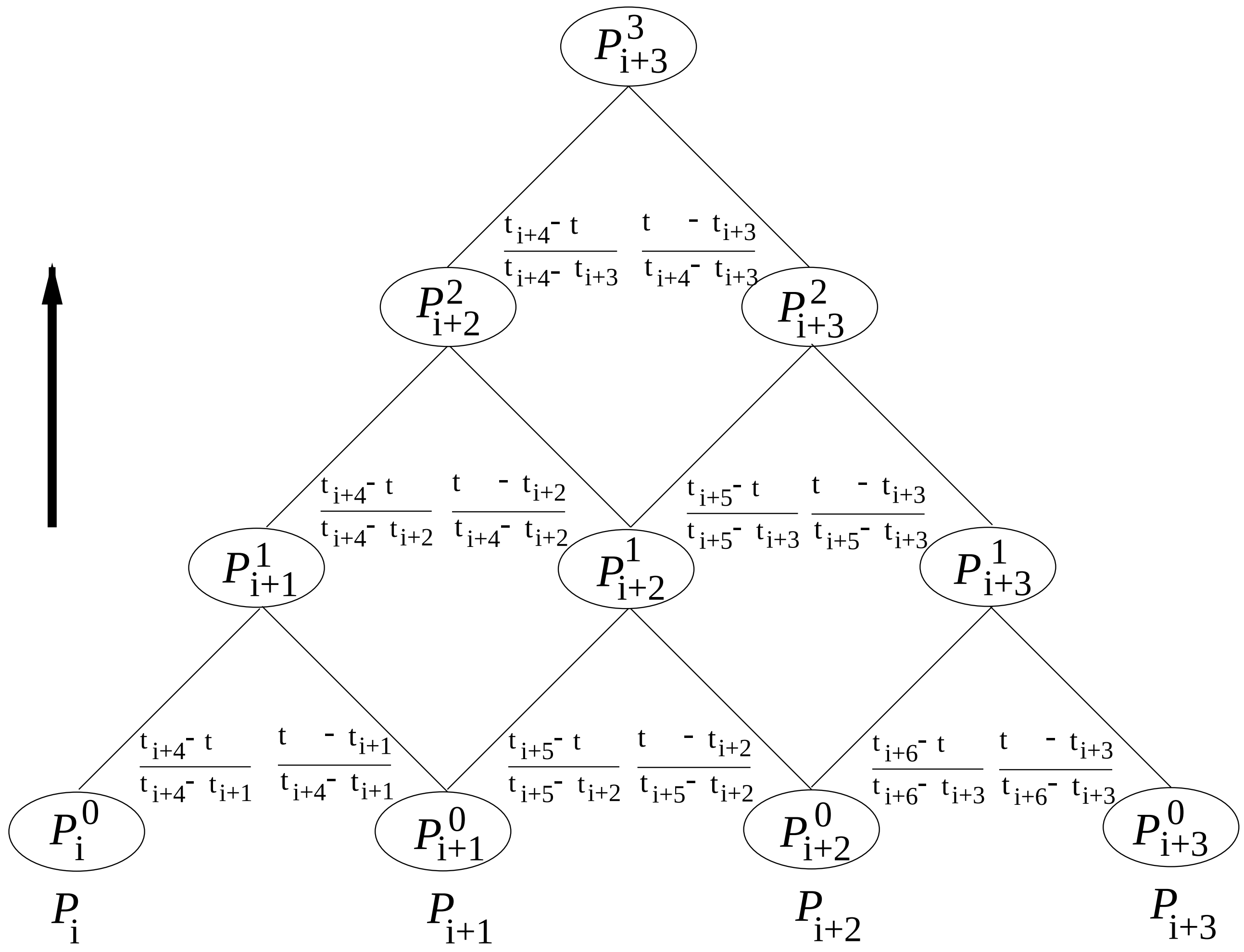
- Nonuniform B-splines (NUBS) generalize this construction
- A B-spline, $B_i^d(t)$, is a piecewise polynomial:
  - each of its segments is of degree $\leq d$
  - it is defined for all t
  - its segmentation is given by $knots\ t = t_0 \leq t_1 \leq \cdots \leq t_N$
  - it is zero for $T < T_i$ and $T > T_{i+d+1}$
  - it may have a discontinuity in its $d - k + 1$ derivative at $t_j \in \{t_i, \ldots, t_{i+d+1}\}$, if t_j has multiplicity k
  - it is nonnegative for $t_i < t < t_{i+d+1}$
  - $B_i^d(t) + \cdots + B_{i+d}(t) = 1$ for $t_{i+d} \leq t < t_{i+d+1}$, and all other $B_j^d(t)$ are zero on this interval
  - Bézier blending functions are the special case where all knots have multiplicity $d + 1$

Example (Quadratic):



Evaluation:

- There is an efficient, recursive evaluation scheme for any curve point
- It generalizes the triangle scheme (deCasteljau) for Bézier curves
- Example (for cubics and $t_{i+3} \leq t < t_{i+4}$):



P^3_{i+3}
$\frac{t_{i+4} - t}{t_{i+4} - t_{i+3}}$
$\frac{t - t_{i+3}}{t_{i+4} - t_{i+3}}$
P^2_{i+2}
P^2_{i+3}
$\frac{t_{i+4} - t}{t_{i+4} - t_{i+2}}$
$\frac{t - t_{i+2}}{t_{i+4} - t_{i+2}}$
$\frac{t_{i+5} - t}{t_{i+5} - t_{i+3}}$
$\frac{t - t_{i+3}}{t_{i+5} - t_{i+3}}$
P^1_{i+1}
P^1_{i+2}
P^1_{i+3}
$\frac{t_{i+4} - t}{t_{i+4} - t_{i+1}}$
$\frac{t - t_{i+1}}{t_{i+4} - t_{i+1}}$
$\frac{t_{i+5} - t}{t_{i+5} - t_{i+2}}$
$\frac{t - t_{i+2}}{t_{i+5} - t_{i+2}}$
$\frac{t_{i+6} - t}{t_{i+6} - t_{i+3}}$
$\frac{t - t_{i+3}}{t_{i+6} - t_{i+3}}$
P^0_i
P^0_{i+1}
P^0_{i+2}
P^0_{i+3}
P_i
P_{i+1}
P_{i+2}
P_{i+3}