Computing Form Factors

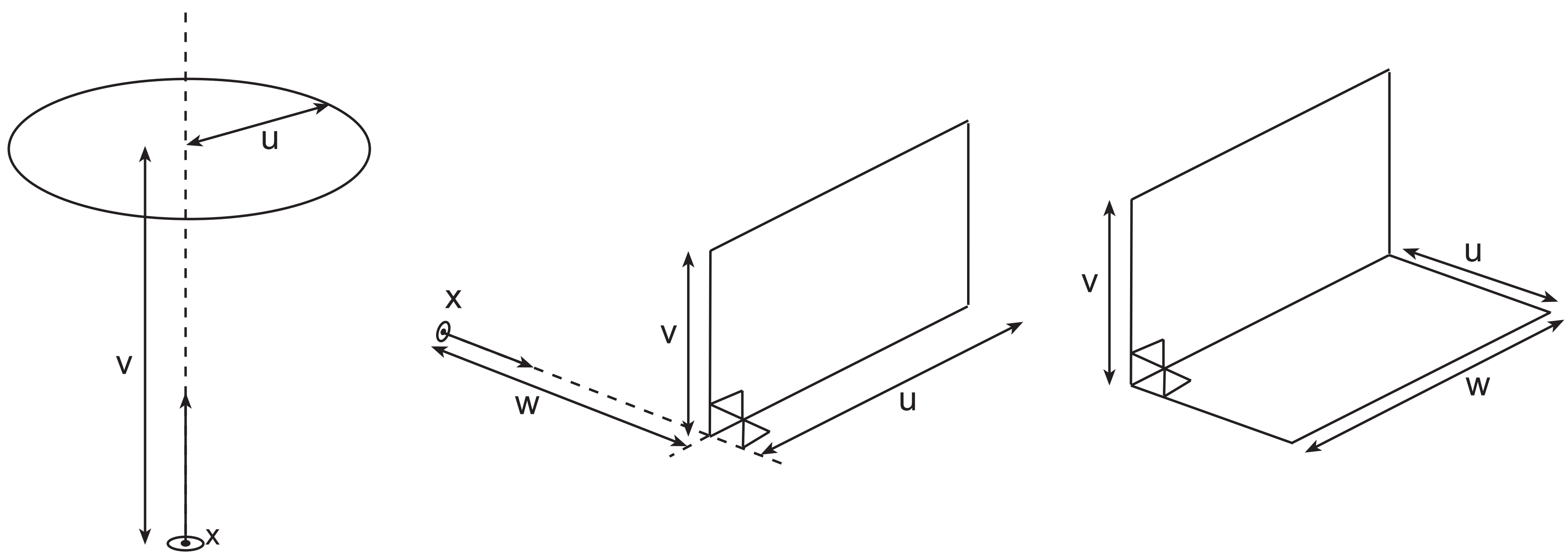
I. Fully Visible

When two patches, P_i and P_j, are fully visible to each other, the form factor has the following expression, obtained by setting the visibility term to 1

$$F_{i,j} = \frac{1}{A_i} \int_{x \in P_i} \int_{y \in P_j} \frac{\cos\theta \cos\theta'}{\pi r^2} dy dx$$

(a) *Direct Integration*

The integral can only be computed directly for very simple arrangements. In most cases these integrals are obtained between primitive shapes such as axis-aligned rectangles, discs, and cylinders. However, it should be noted that even the conceptually simple case of two isolated polygons in space results in a very complex expression for the form factor.



(a) (b) (c)

Some examples for which the form factor integral can be computed analytically.

Figure (a) shows a "pointwise" form factor from a point x to a disc perpendicular to the direction joining x to its center:

$$F_{x,\text{disc}} = \frac{u^2}{u^2 + v^2}, \qquad X = \frac{u}{w} \qquad Y = \frac{v}{w}.$$

Figure (b) shows a form factor between a surface element at point x and a parallel rectagle perpendicular to the direction joining x to one of its corners:

$$F_{x,\text{rect}} = \frac{1}{2\pi}\left[\frac{X}{\sqrt{1+X^2}}\tan^{-1}\left(\frac{Y}{\sqrt{1+X^2}}\right) + \frac{Y}{\sqrt{1+Y^2}}\tan^{-1}\left(\frac{X}{\sqrt{1+Y^2}}\right)\right].$$

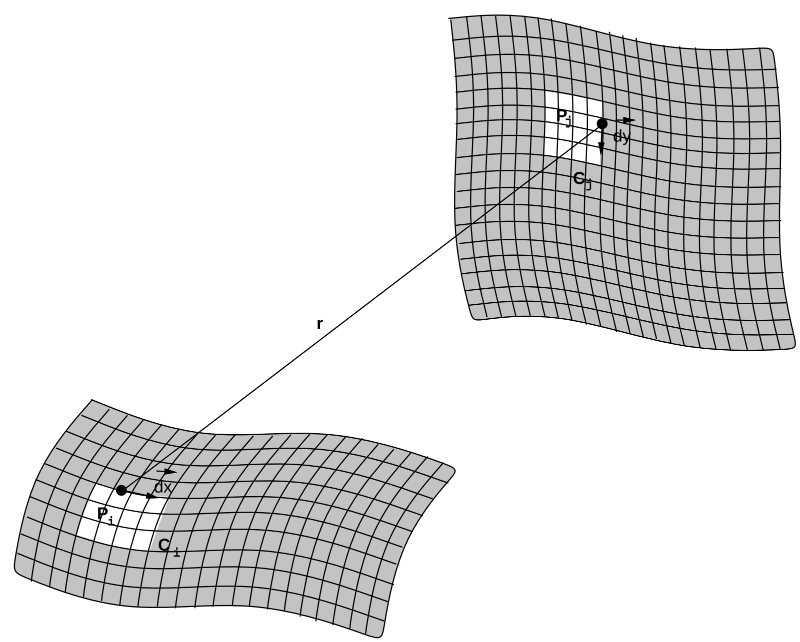
Figure (c) shows a form factor between two perpendicular rectangles having a common edge:

$$\begin{aligned}F_\perp = {}& \frac{1}{\pi X}\left\{X\tan^{-1}\left(\frac{1}{X}\right) + Y\tan^{-1}\left(\frac{1}{Y}\right) - \sqrt{X^2+y^2}\tan^{-1}\frac{1}{\sqrt{X^2+Y^2}}\right\} \\ & + \frac{1}{4\pi X}\left\{\ln\left[\frac{(1+X^2)(1+Y^2)}{1+X^2+Y^2}\right] + X^2\ln\left[\frac{X^2(1+X^2+Y^2)}{(1+X^2)(X^2+Y^2}\right]\right. \\ & \left. + Y^2\ln\left[\frac{Y^2(1+X^2+Y^2)}{(1+Y^2)(X^2+Y^2}\right]\right\}.\end{aligned}$$

(b) *Contour Integration*

Using Stokes' theorem, the area interals can be transformed into contour integrals over the boundaries of the two patches.

$$F_{ij} = \frac{1}{2\pi A_i} \oint_{C_i} \oint_{C_j} \ln r \; d\vec{x} \cdot d\vec{y}.$$



Contours used to integrate teh form factor.

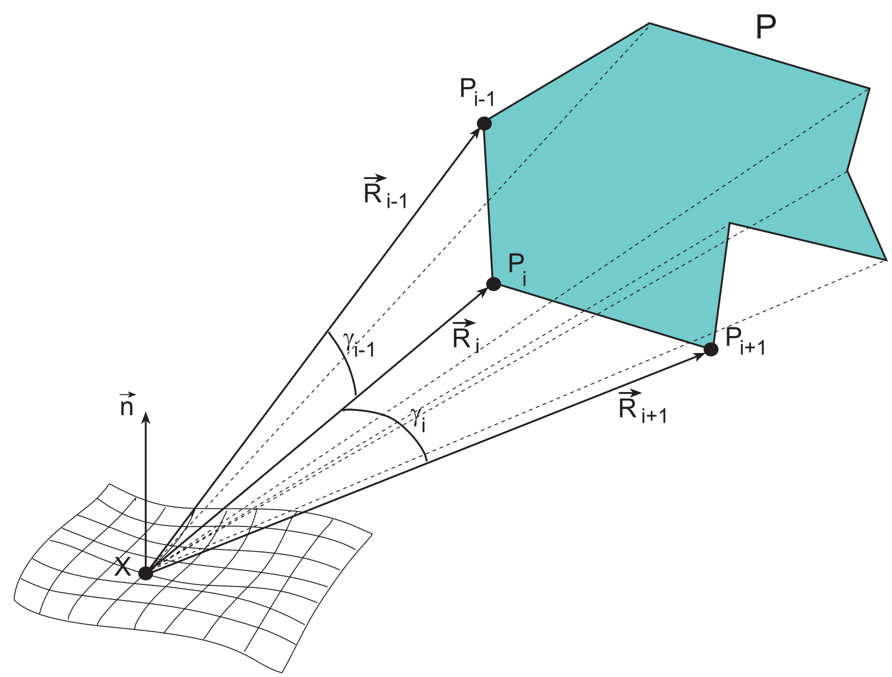
(c) *Angle Formula*

In the case of polygonal patches, contour integration can be carried out explicitly along polygon edges. When estimating the radiosity at a particular point x on a surface, the "point-to-point" form factors are used, and can be computed using the following formula

$$F_{x,P} = \frac{1}{2\pi} \sum_{g \in G} \vec{n} \cdot \vec{\Gamma}_g,$$

where $\vec{n}$ is the normal vector of the surface at point x, G is the set of vertices of the polygon, $\vec{R}_g$ is a vector from x to vertex number g of patch P, and $\vec{\Gamma}_g$ is a vector oriented in the direction of the cross product $\vec{R}_g \times \vec{R}_{g+1}$, with magnitude equal to the angle γ_g.

The formula is obtained by a single contour integral computed around patch P as a sum of integrals along each edge of P.



Analytic formula for the unoccluded form factor between a point and a polygonal patch.

(d) *Form Factor Algebra*

In the situations depicted in the following figure, a simple exchange of integration variables shows that the double integrals in the form factor expressions for F_{12} and F_{34} are equal. Thus

$$A_1 F_{12} = A_3 F_{34}. \qquad \text{(Yamauti principle)}$$

This way of using the symmetry between two form factor integrals is known as the *Yamauti principle* or *reciprocity theorem*. It can be used to compute unknown form factors from known ones, a process commonly referred to as *form factor algebra*.

Simple Properties of Form Factors:

1) Reciprocity: $\forall i, j \quad A_i F_{ij} = A_j F_{ji}$
2a) Additivity: if $P_l = P_j \cup P_k$, then $F_{il} = F_{ij} + F_{ik}$
2b) if patches overlap: $P_m = P_j \cap P_k \neq \emptyset$, then $F_{il} = F_{ij} + F_{ik} - F_{im}$
3) Non-distributivity: $F_{li} \neq F_{ji} + F_{ki}$
4) Partition of Unity: $\forall i \; \sum_{j=1}^{N} F_{ij} = 1$

Using the simple properties of the form factor, the following transformations can be applied to the known form factor F_{87}:

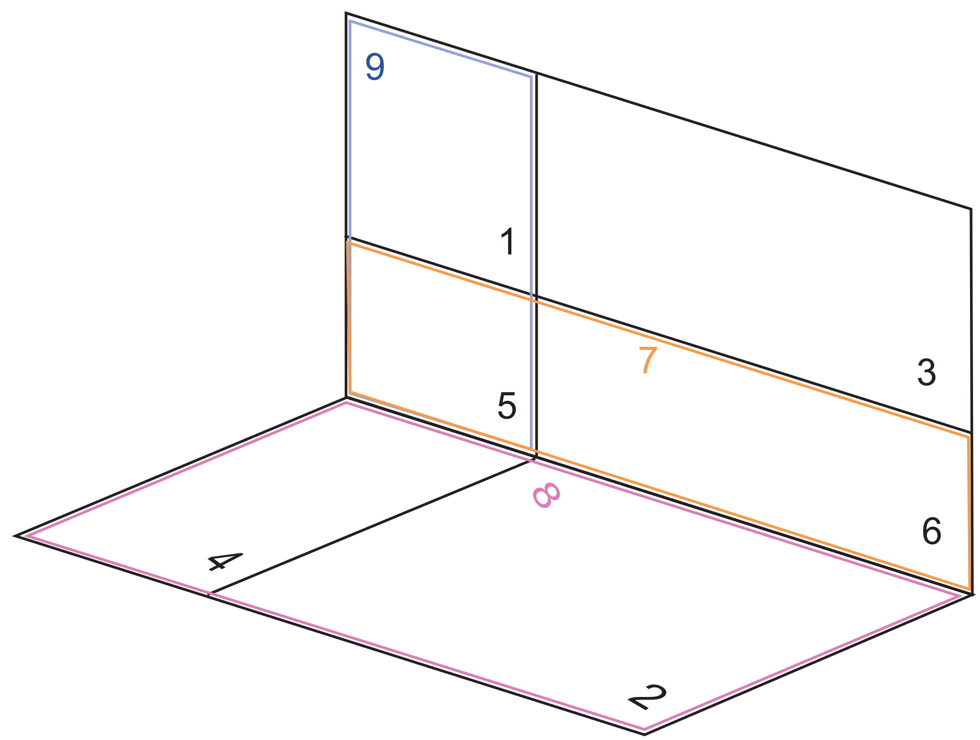
$$\begin{aligned} F_{87} &= F_{85} + F_{86} \\ &= \frac{A_5}{A_8} F_{58} + \frac{A_6}{A_8} F_{68} \\ &= \frac{A_5}{A_8}(F_{54} + F_{52}) + \frac{A_6}{A_8}(F_{62} + F_{64}). \end{aligned}$$

This equation is easily solved for F_{52} to yield

$$F_{52} = \frac{A_8 F_{87} - A_5 F_{54} - A_6 F_{62}}{A_5}.$$

The form factor between the union of patches 1 and 5, called patch 9, and patch 2, can be computed in the same manner. The final result is obtained by combining these two factors to yield

$$F_{12} = \frac{A_9 F_{92} - A_5 F_{52}}{A_1}.$$



An example of form factor algebra.
Larger patches are defined by $7 = 5 \cup 6$, $8 = 4 \cup 2$, and $9 = 1 \cup 5$.

II. Projection Methods

When two surface patches are distant from each other, relative to their size, the inner integral in the form factor definition

$$F_{i,j} = \frac{1}{A_i} \int_{x \in P_i} \int_{y \in P_j} \frac{\cos\theta \cos\theta'}{\pi r^2} V(x,y) dy dx$$

varies very little across the surface of patch P_i, so that the effect of the outer integral is merely a multiplication by 1. In such a case, the form factor can be computed as that from a point (a differential area) to a finite area.

When the above assumption does not hold, one can imagine subdividing patch P_i into smaller subpatches to enforce the condition, and then combining the results using the area-weighted average. In this way the problem of computing form factors between distant finite areas is reduced to the somewhat simpler problem of computing form factors between points and finite areas.

Nusselt's Analogy

An alternate formulation of the form factor can be obtained by looking back at the definition of the incident flux density H on a surface:

$$H(x) = \int_\Omega L_i(x, \theta, \phi) \cos\theta d\omega.$$

Comparing the simple energy balance equation

$$B(x) = E(x) + \rho_d(x) H(x)$$

with

$$B(x) = E(x) + \rho_d(x) \sum_{j=1}^{N} B_j \int_{y \in P_i} \frac{\cos\theta \cos\theta'}{\pi r^2} V(x, y) dy,$$

we obtain an expression for the pointwise form factor from point x to a given patch P

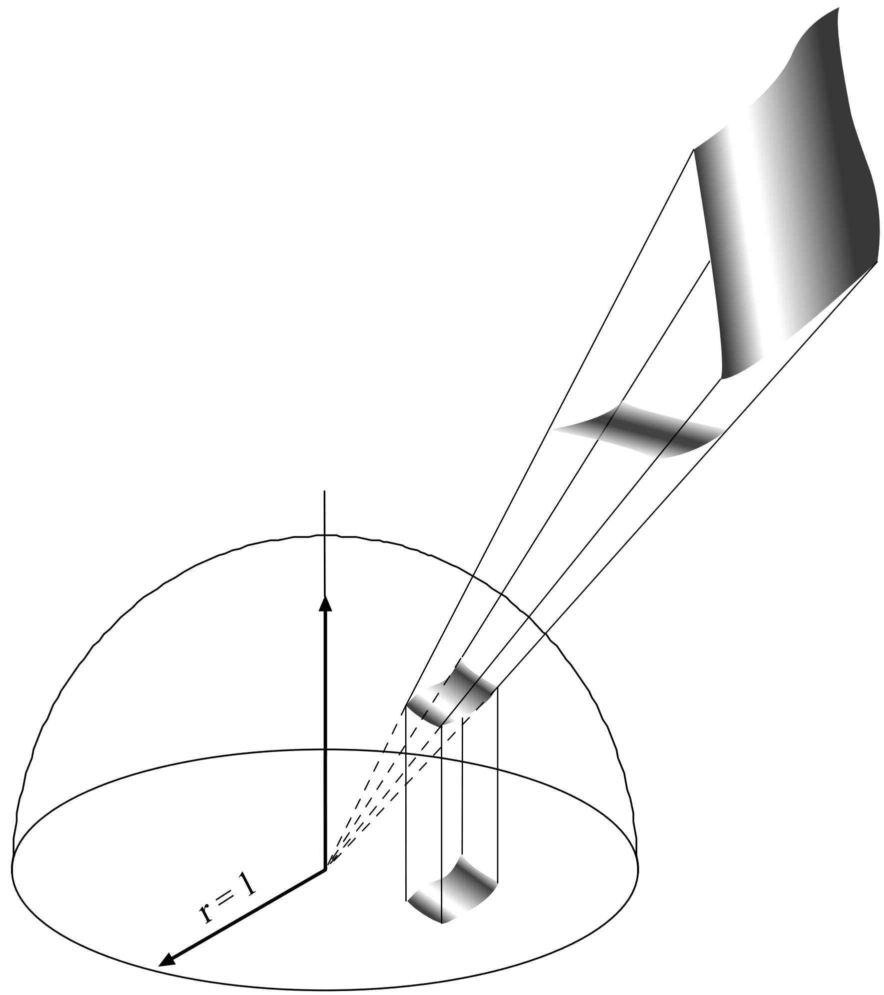
$$F_{x,P} = \frac{1}{\pi} \int_{\Omega_P} \cos\theta \, d\omega,$$

which is the integral over the solid angle subtended by patch P. Here Ω_P represents the set of directions that allow x to see a point on P.

$F_{x,P}$ is the fraction of the area of the unit disc in the base plane obtained by projecting surface patch P onto the unit hemisphere Ω centered at point x, and then orthogonally down onto the base plane.

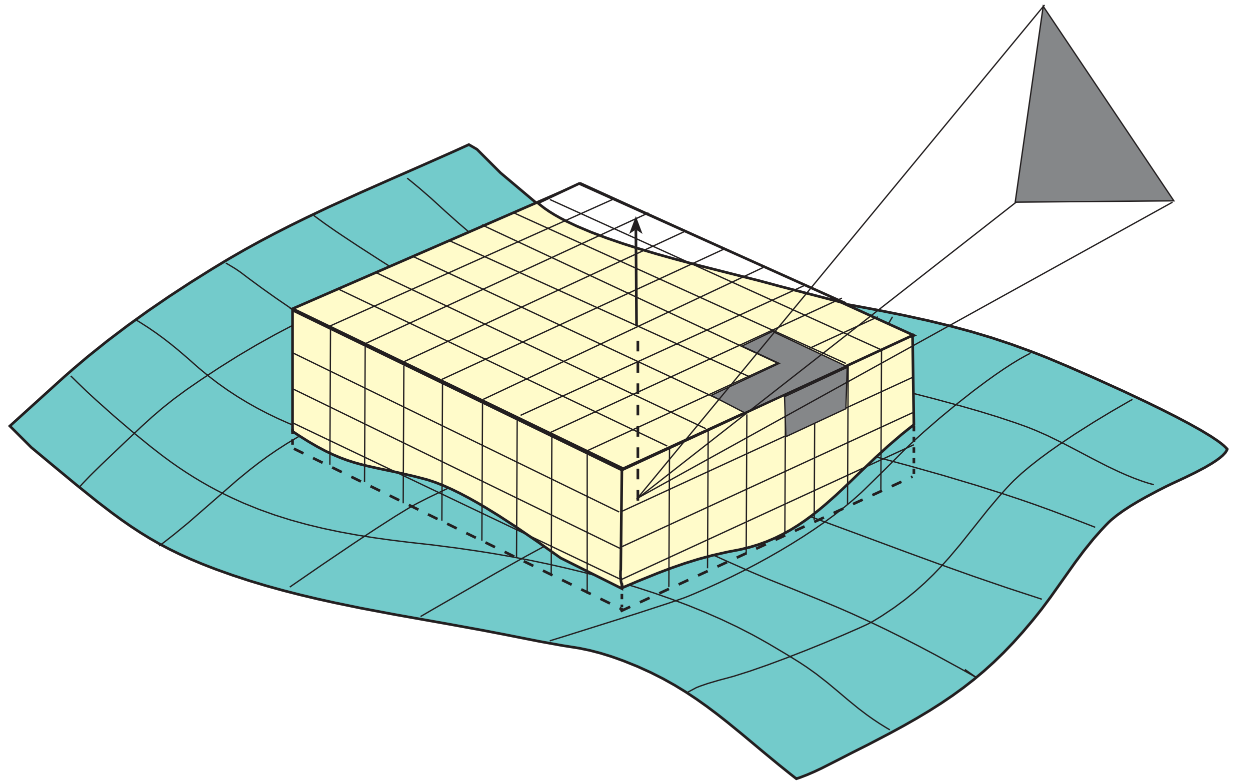
Here the "base plane" is simply the tangent plane of the surface at point x.

It follows from the above definition that two surfaces sharing the same projection on Ω share the same form factor from x. In other words, the form factor from x to a given surface depends on only the projection of that surface onto the hemisphere Ω. The following figure shows Nusselt's construction.



Nusselt's analogy

The Hemi-cube



The hemi-cube.

The faces of the hemi-cube are easily discretized into a number of square cells, for which a delta-form factor ΔF can be computed analytically. In practice an approximation can be used safely since the area ΔA of a cell is usually very small compared to unity.

- For a cell on the top face of the hemi-cube:

$$\Delta F = \frac{\Delta A}{\pi(1 + x^2 + y^2)^2}.$$

- For a cell on the side face of the hemi-cube:

$$\Delta F = \frac{z \Delta A}{\pi(1 + x^2 + y^2)^2}.$$

Each face of the hemi-cube is used in turn as a projection surface to determine visibility, by means of an *item buffer*. An item buffer is an array of patch identifiers — one for each hemi-cube cell – that records the visible surface for the direction of each cell. It is constructed by the following the common z-buffer strategy:

- For each cell intersected by the projection, the distance to the projected patch is computed and compared to the distance previously recorded for that cell. Each patch is given a number from 1 to N. If the patch being projected is closer than the recorded distance, its number is recorded in the item buffer and its distance in the z-bufer. The sole difference with the usual z-buffer technique is that the patch number of the nearest patch is kept rather than the patch color. The closest surface patch visible through each cell is thus determined, and after all patches have been projected, the item buffers contain all the relevant visibility information. The accuracy of this information naturally depends on the hemi-cube resolution.

- The form factor to all surfaces are then computed at once. Each hemi-cube cell contributes its delta-form factor to the form factor with its visible patch. The form factor to a given patch is obtained by summing the delta-form factors corresponding to all the cells covered by the projection of the patch. In other words, if $C(j)$ is the set of hemi-cube cells through which patch P_j is visible, then

$$F_{x,P_j} = \sum_{q \in C(j)} \Delta F_q.$$

Pros & Cons

- Hemi-cube is its limited resolution. If N_c hemi-cube cells, no more than N_c patches will have a non-zero form factor from any given point since only N_c delta-form factors can be assigned. This quantization problem can lead to severe inaccuracies.
- Hemi-cube is appealing because of its simplicity and speed. Z-buffer (hardware) can be used to compute contribution for faces of the hemi-cube quickly at the cost of limited precision.